Existence, non existence et multiplicité d'ondes stationnaires normalisées pour quelques équations non linéaires elliptiquesExistence, non-existence and multiplicity of normalized standing waves for some nonlinear elliptic equations

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Existence, non-existence and multiplicity of normalized standing waves for some nonlinear elliptic equations

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Résumé

Résumé

Dans cette thèse, nous étudions l'existence, non existence et multiplicité des ondes stationnaires avec les normes prescrites pour deux types d'équations aux dérivées partielles non linéaires elliptiques découlant de différents modèles physiques. La stabilité orbitale des ondes stationnaires est également étudiée dans certains cas. Les principales méthodes de nos preuves sont des arguments variationnels. Les solutions sont obtenues comme points critiques de fonctionnelle associée sur une contrainte.

La thèse se compose de sept chapitres. Le Chapitre 1 est l'introduction de la thèse. Dans les Chapitres 2 à 4, nous étudions une classe d'équations de Schrödinger-Poisson-Slater non linéaires. Nous établissons dans le Chapitre 2 des résultats optimaux non existence de solutions d'énergie minimale ayant une norme L^2 prescrite. Dans le Chapitre 3, nous montrons un résultat d'existence de solutions L^2 normalisées, dans une cas où la fonctionnelle associée n'est pas bornée inférieurement sur la contrainte. Nos solutions sont trouvées comme des points de selle de la fonctionnelle, mais ils correspondent à des solutions d'énergée minimale. Nous montrons également que les ondes stationnaires associées sont orbitalement instables. Ici, puisque nos points critiques présumés ne sont pas des minimiseurs globaux, il n'est pas possible d'utiliser de façon systématique les méthodes de compacité par concentration développées par P. L. Lions. Ensuite, dans le Chapitre 4, nous montrons que sous les hypothèses du Chapitre 3, il existe une infinité de solutions ayant une norme L^2 prescrite. Dans les deux chapitres suivants, nous étudions une classe d'équations de Schrödinger quasi-linéaires. Des résultats optimaux non existence de solutions d'énergie minimale sont donnés dans le Chapitre 5. Dans le Chapitre 6, nous prouvons l'existence de deux solutions positives ayant une norme donnée. L'une d'elles, relativement à la contrainte L^2, est de type point selle. L'autre est un minimum, soit local ou global. Le fait que la fonctionnelle naturelle associée à cette équation n'est pas bien définie nécessite l'utilisation d'une méthode de perturbation pour obtenir ces deux points critiques. Enfin, au Chapitre 7, nous mentionnons quelques questions que cette thèse a soulevées.

Mots-clefs

ondes stationnaires, norme L^2 prescrite, non existence optimale, minimiseurs globaux ou locaux, multiplicité des solutions normalisées, forte instabilité par explosion, méthodes variationnelles, arguments de perturbation, équations de Schrödinger-Poisson-Slater, équations de Schrödinger quasi-linéaires.

Existence, non-existence and multiplicity of normalized standing waves for some nonlinear elliptic equations

Abstract

In this thesis, we study the existence, non-existence and multiplicity of standing waves with prescribed norms for two types of nonlinear elliptic partial differential equations arising from various physical models. The orbital stability of the standing waves is also discussed in some cases. The main ingredients of our proofs are variational arguments. The solutions are found as critical points of an associated functional on a constraint.

The thesis consists of seven chapters. Chapter 1 is the Introduction of the thesis. In Chapters 2 to 4, we study a class of nonlinear Schrödinger-Poisson-Slater equations. We establish in Chapter 2 sharp non-existence results of least energy solutions having a prescribed L^2-norm. In Chapter 3 we prove an existence result for L^2-normalized solutions, in a situation where the associated functional is unbounded from below on the constraint. Our solutions are found as saddle points of the functional but they correspond to least energy solutions. We also prove that the associated standing waves are orbitally unstable. Here a key feature is that, since our suspected critical points are not global minimizers, it is not possible to use in a standard way the machinery of compactness by concentration developed by P. L. Lions. Then, in Chapter 4, we prove that under the assumptions of Chapter 3, there do exist infinitely many solutions having a prescribed L^2-norm. In the following two chapters, we investigate a class of quasi-linear Schrödinger equations. Sharp non-existence results of least energy solutions are given in Chapter 5. In Chapter 6 we prove the existence of two positive solutions having a given norm. One of them, is relative to the L^2-norm constraint, of saddle point type. The other one is a minimum, either local or global. The fact that the natural functional associated with this equation is not well defined requires the use of a perturbation approach to obtain these two critical points. Finally, in Chapter 7 we mention some questions that this thesis has raised.

Keywords

standing waves, prescribed L^2-norm, sharp non-existence, global or local minimizers, multiplicity of normalized solutions, strong instability by blow up, variational methods, perturbation arguments, Schrödinger-Poisson-Slater equations, quasi-linear Schrödinger equations.

AMS Subject Classification (2000)

35J50, 35Q41, 35Q55, 37K45.

Notations

We list below the notations and definitions that we use throughout the thesis.

(1) $\mathbb{C}, \mathbb{R}, \mathbb{Z}, \mathbb{N}$ denote respectively the set of complex numbers, real numbers, integers and positive integers.

(2) i denotes the imaginary unit.

(3) Let $Re\ z$, $Im\ z$ denote respectively the real and imaginary parts of a complex number $z \in \mathbb{C}$.

(4) $\mathbb{R}^N$ denotes the N-dimensional real Euclidean space, $\mathbb{R} = \mathbb{R}^1$, and a typical point in $\mathbb{R}^N$ is $x = (x_1, x_2, ..., x_N)$.

(5) $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}^- := \{x \in \mathbb{R} \mid x < 0\}$.

(6) Denote $u^+(x)(u^-)$ the positive (negative) part of the function $u(x)$, namely $u^+ = \max\{u, 0\}, u^- = \max\{-u, 0\}$. In particular, $u = u^+ - u^-$ and $|u| = u^+ + u^-$.

(7) $B(x, r)$ denotes a closed ball with center x, radius $r > 0$.

(8) The sign function is defined as

$$sgn(x) := \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

(9) $\chi_E(x) := \begin{cases} 1, & \text{if } x \in E, \\ 0, & \text{if } x \notin E. \end{cases}$, denotes the characteristic function of the set E.

(10) The convolution of the functions f, g is denoted by

$$f * g.$$

(11) ∇u denotes the gradient of a differentiable function u, namely $\nabla u = (\partial_{x_1} u, ..., \partial_{x_N} u)$. When $N = 1$, we replace ∇ by d/dx.

(12) Δ denotes the Laplacian operator, $\Delta := \Sigma_{i=1}^N \partial^2/\partial_{x_i}^2$.

(13) Let Ω be a bounded domain in $\mathbb{R}^N$. Then $\partial\Omega$ denotes the boundary of Ω, and $\bar{\Omega} := \Omega \cup \partial\Omega$ denotes the closure of Ω.

(14) *supp* u denotes the support of a function u defined in $\mathbb{R}^N$, that is

$$supp\ u = \overline{\{x \in \mathbb{R}^N \mid u(x) \neq 0\}}.$$

(15) u^* denotes the Schwartz symmetrization of a function u.

(16) For any $1 \le q < +\infty$, we write $L^q(\mathbb{R}^N)$ as the usual Lebesgue space endowed with the norm

$$\|u\|_q^q := \int_{\mathbb{R}^N} |u|^q dx,$$

and $W^{1,q}(\mathbb{R}^N)$ the usual Sobolev space endowed with the norm

$$\|u\|_{W^{1,q}} := \|\nabla u\|_q + \|u\|_q.$$

In particular, we denote briefly $\|\cdot\| := \|\cdot\|_{W^{1,2}}$.

(17) $W_r^{1,q}(\mathbb{R}^N)$ denotes the radial symmetric subspace of $W^{1,q}(\mathbb{R}^N)$.

(18) $L^\infty(\mathbb{R}^N)$ denotes the set of almost everywhere bounded functions defined in $\mathbb{R}^N$.

(19) 2^* denotes the critical exponent of the Sobolev embedding, namely

$$2^* = \frac{2N}{N-2} \text{ if } N \ge 3 \text{ and } 2^* = +\infty \text{ if } N = 1, 2.$$

(20) $\mathcal{D}^{1,2}(\mathbb{R}^N) := \{u \in L^{2^*}(\mathbb{R}^N) : \ \nabla u \in L^2(\mathbb{R}^N)\}$.

(21) X^* denotes the dual of a Banach space X.

(22) Let U, V, W be three open subsets of $\mathbb{R}^N$. We write

$$V \subset\subset U$$

if $V \subset \bar{V} \subset U$ and $\bar{V}$ is compact, and say V is compactly contained in U.

(23) For any $1 \le p < \infty$, $L^p_{loc}(\mathbb{R}^N) := \{u : \mathbb{R}^N \to \mathbb{R} \mid u \in L^p(\Omega) \text{ for each } \Omega \subset\subset \mathbb{R}^N\}$.

(24) In a metric space (X, ρ) with the metric ρ, we denote by $dist\{U, V\}$ the distance between two sets $U \subset X$ and $V \subset X$, namely

$$dist\{U, V\} := \inf_{x \in U, y \in V} \rho(x, y).$$

(25) For convenience, we denote for $u \in W^{1,2}(\mathbb{R}^3)$ the following quantities

$$A(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx, \quad B(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy$$

$$C(u) := \int_{\mathbb{R}^3} |u|^p dx, \quad D(u) := \int_{\mathbb{R}^3} |u|^2 dx.$$

(26) Throughout the thesis, we denote by $C > 0$ various positive constants which may vary from one line to another but not affect the analysis of the problem.

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Chapter 1

Introduction

This thesis is devoted to the study of two types of nonlinear elliptic partial differential equations, namely a class of nonlinear Schrödinger-Poisson-Slater equations and a class of quasi-linear Schrödinger equations. These equations originate from various models in theoretical or applied physics. Due to their sound physical backgrounds as well as the mathematical challenges that they present, the study of these equations has been extremely active during the last decade. In this thesis, we are concerned with the existence, non-existence and multiplicity of standing waves for the two types of equations. In some cases, we also establish the orbital instability of the standing waves. A common feature of our results is that we deal with solutions having a prescribed L^2-norm. The study of normalized solutions is important from the point of view of physics. The main ingredients of our proofs are variational arguments. The solutions are indeed found as critical points of an associated functional on a constraint.

In addition to this Introduction, the thesis consists of other six chapters. In Chapters 2 to 4, we study a class of nonlinear Schrödinger-Poisson-Slater equations. We establish in Chapter 2 sharp non-existence results of least energy solutions having a prescribed L^2-norm. In Chapter 3 we prove an existence result for L^2-normalized solutions, in a situation where the associated functional is unbounded from below on the constraint. Our solutions are found as saddle points of the functional but they correspond to least energy solutions. We also prove that the associated standing waves are orbitally unstable. Here a key feature is that, since our suspected critical points are not global minimizers, it is not possible to use in a standard way the machinery of compactness by concentration developed by P. L. Lions [83]. Then, in Chapter 4, we prove that under the assumptions of Chapter 3 there do exist infinitely many normalized solutions. In the following two chapters, we investigate a class of quasi-linear Schrödinger equations. Sharp non-existence results of least energy solutions having a prescribed L^2-norm, are given in Chapter 5. In Chapter 6 we prove the existence of two positive solutions having a given norm. One of them, is relative to the L^2-norm constraint, of saddle point type. The other one is a minimum, either local or global. The fact that the natural functional associated with this equation is not well defined requires the use of a perturbation approach to obtain these two critical points. Finally in Chapter 7 we mention some of the questions that this thesis has raised.

All the material of the thesis correspond to some already published works or to some preprints. In particular Chapters 2 and 5 correspond to the paper [65] in collaboration with L. Jeanjean. Chapter 3 corresponds to the work [16] with J. Bellazzini (Univ. Sassari - Italy) and L. Jeanjean. Chapter 4 to [90] and Chapter 6 to a work with L. Jeanjean and Z.-Q. Wang (Univ. Logan - USA) [66].

1.1 Schrödinger-Poisson-Slater Equations

Consider the time-dependent Hartree-Fock system in $\mathbb{R}^3$:

$$i\partial_t\psi_j + \Delta\psi_j - \kappa(|x|^{-1} * \rho)\psi_j + (V_{ex} \bullet \psi_j) = 0,\ j \in \mathbb{N}, \tag{1.1.1}$$

where $\kappa \in \mathbb{R}$, $\psi_j = \psi_j(t,x) :\ \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{C}$ is the j^{th} component of a vector function ψ, ρ is the local density, given by $\rho := \sum_{k\in\mathbb{N}} \theta_k|\psi_k|^2$ with $\{\theta_k\} \subset \mathbb{R}$ being a non-negative sequence satisfying $\sum_{k\in\mathbb{N}} \theta_k = 1$, and $(V_{ex} \bullet \psi_j)$ is called by the Fock potential. In particular, when $1 \le j \le N$ for $N \in \mathbb{N}$, it is given precisely that

$$(V_{ex} \bullet \psi_j) := \sum_{k=1}^{N} \int_{\mathbb{R}^3} |x-y|^{-1}\psi_k(x)\bar{\psi}_k(y)\psi_j(y)dy.$$

This system has been widely used in the study of systems of many particles in Atomic Physics and Quantum Mechanics. In (1.1.1), the term $\kappa(|x|^{-1} * \rho)$ is called by Hartree potential, formally similar with the fundamental solution of a Poisson equation (see [21, 102]). The sign of the parameter $\kappa \in \mathbb{R}$ depends on the type of interaction between the particles: $\kappa > 0$ in the repulsive case and $\kappa < 0$ in the attractive case. However, in this system, the Fock potential $(V_{ex} \bullet \psi_j)$, especially when $N > 0$ is large, is too complex to allow practical calculations. Thus J. Slater [106] proposed a simple approximation of this term in the form

$$(V_{ex} \bullet \psi_j) \simeq \eta\rho^{1/3}\psi_j,\ 1 \le j \le N,$$

where $\eta > 0$ is a positive constant. Then the Hartree-Fock system is simplified as

$$i\partial_t\psi_j + \Delta\psi_j - \kappa(|x|^{-1} * \rho)\psi_j + \eta\rho^{1/3}\psi_j = 0,\ \forall\ 1 \le j \le N. \tag{1.1.2}$$

The equations (1.1.2) are known as the Schrödinger-Poisson-Slater system. This system is an important model used for the study of quantum transport in semiconductor devices (see [6]).

Particularly, when $\theta_1 = 1$ and $\theta_k = 0$ for $k \ge 2$, the system (1.1.2) is reduced to the *single-state-case*, which serves as a local single-particle approximation of the Hartree-Fock system. When $\theta_k = 1/K$ if $1 \le k \le K$ for some $K \le N$, and $\theta_k = 0$ otherwise, (1.1.2) appears in the case of K coupled equations. Such a model stems from the density functional theory, having been used in the Molecular Quantum Chemistry. See for instance [73]. In addition, when $\theta_k \in l^1$, (1.1.2) describes the *mixed-state-case*. In that direction, we refer to a result established by F. Castella [33], concerning the existence and uniqueness of L^2 solution for (1.1.2) as $\eta = 0$.

In this thesis, we are concerned with the following single-state-case of the Schrödinger-Poisson-Slater equations:

$$i\partial_t\psi + \Delta\psi - (|x|^{-1} * |\psi|^2)\psi + |\psi|^{p-2}\psi = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \tag{1.1.3}$$

where $p \in (2,6)$. Here we have generalized the term $|\psi|^{2/3}\psi$ by the local nonlinear term $|\psi|^{p-2}\psi$. As we shall see the results we shall obtain on (1.1.3) will depend strongly on the value of $p \in (2,6)$. Some particular cases have a direct physical interpretation, for example the value $p = \frac{10}{3}$ gives rise to the so-called Dirac correction (see e.g. [102]). Observe that when $p < 2$ or $p > 6$ the embedding of $H^1(\mathbb{R}^3)$ into $L^p(\mathbb{R}^3)$ is lost. That is why we restrict our analysis to the range $p \in (2,6)$. Note also that in the study of the attractive case, namely $\kappa < 0$ in (1.1.2), one is allowed to introduce symmetric rearrangements that contribute to simplify some computations, see e.g. [76, 80]. By contrast, the repulsive case (1.1.3) is more difficult and thus we shall deal only with it. The modified system (1.1.3), has also a wide range of applications in other fields, for instance in the Thomas-Fermi theory (see [78, 79]). Such kinds of approximations are usually called X^α-*approaches* (here $\alpha := p$), see e.g. [6, 21, 102]. In view of these, over the past few decades, an extensive body of studies have contributed to the problem (1.1.3). The pertinent literature is rather wide and rich. We restrict ourselves to cite [4, 43, 70, 71, 72, 99, 100, 102] and their references given there. In the cited references, a central issue is the study of standing wave solutions of (1.1.3).

By standing waves of (1.1.3), we mean solutions of the form

$$\psi(t,x) = e^{-i\lambda t}u(x),$$

where $\lambda \in \mathbb{R}$ is a parameter. Then the function $u(x)$ satisfies the stationary Schrödinger-Poisson-Slater equation

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^3. \tag{E_λ}$$

The equation (E_λ) also was proposed by V. Benci and D. Fortunato [17] as a model to describe the interaction of a quantum particle with an electromagnetic field.

To find solutions of (E_λ), mainly two approaches have been developed. A first one is to consider $\lambda \in \mathbb{R}$ as a fixed parameter and then to search for a $u(x)$ solving (E_λ). Then solutions of (E_λ) correspond to critical points of the functional defined in $H^1(\mathbb{R}^3)$,

$$F_\lambda(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx. \tag{1.1.4}$$

Along this line, the existence, non-existence and multiplicity of solutions have been extensively studied by many authors. See e.g. [4, 7, 43, 45, 62, 63, 70, 71, 72, 99, 104, 108, 112] and their references therein.

An alternative approach motivated in particular by the fact that physicists are interested in “normalized solutions” is to search for solutions of (E_λ), having a prescribed L^2-norm. More precisely, for given $c > 0$, one looks to

$$(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R} \;\text{ solutions of } (E_\lambda) \text{ with } \|u_c\|_2^2 = c.$$

In this case, a solution $u_c \in H^1(\mathbb{R}^3)$ of (E_λ) can be obtained as a constrained critical point of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx, \tag{1.1.5}$$

on the constraint

$$S(c) := \left\{u \in H^1(\mathbb{R}^3) :\; \|u\|_2^2 = c, c > 0\right\}. \tag{1.1.6}$$

The parameter $\lambda_c \in \mathbb{R}$, in this situation, can not fixed any more and it appears as a Lagrange parameter. Note that for any $p \in (2,6)$, the functional F, defined by (1.1.5), is well defined and C^1 on $S(c)$, see e.g. [99].

In Chapters 2 to 4, we investigate the existence, non-existence, multiplicity and dynamics of standing waves of (1.1.3). The standing waves are obtained as constrained critical points of the functional F on $S(c)$.

1.1.1 Solutions as global minimizers

To find critical points of F on $S(c)$, we first consider the following minimization problem

$$m(c) := \inf_{u \in S(c)} F(u). \tag{1.1.7}$$

Clearly, minimizers of $m(c)$ are critical points of F restricted to $S(c)$, and then solutions of (E_λ) for some $\lambda \in \mathbb{R}$. We observe, see Remark 2.1.3, that for any $c \in (0, \infty)$, $p = \frac{10}{3}$ is a threshold value for $m(c)$. Namely $m(c) \in (-\infty, 0]$ if $p \in (2, \frac{10}{3})$ and $m(c) = -\infty$ if $p \in (\frac{10}{3}, 6)$. When $m(c) > -\infty$, it is possible to search for solutions of (E_λ) as minimizers of $m(c)$. Previous results on that problem had been established in [14, 15, 102, 105]. See also [34, 72] for a closely related problem. We collect these results in the following theorem.

Theorem 1.1.1 ([14, 15, 102, 105])**.** *(1) Let $p \in (2, 3)$. Then there exists a $c_1^* > 0$, such that for all $c \in (0, c_1^*)$, $m(c) < 0$ and it admits a minimizer.*
(2) Let $p \in (3, \frac{10}{3})$. Then there exists a $c_2^* > 0$, such that for all $c \in (c_2^*, \infty)$, $m(c) < 0$ and it admits a minimizer.

We point out that it is still not known if the minimizers of $m(c)$, or at least one of them, are radially symmetric. In that direction we are only aware of the result of V. Georgiev, F. Prinari and N. Visciglia [52] which gives a positive answer when $p \in (2, 3)$ and for $c > 0$ sufficiently small. In this range the critical point is found as a minimizer of $F(u)$ on $S(c)$. Since there may not exist a minimizer which is radially symmetric, it is not possible to restrict the variational problem to the subset of radially symmetric functions. Such restriction would simplify the treatment of the possible loss of compactness by proving a weakly lower semi-continuous property to F.

The main ingredient in the proofs of the results presented in Theorem 1.1.1 is the use of the concentration compactness principle of P. L. Lions [83]. The key point is to show that the minimizing sequences for $m(c)$ are, up to translations, pre-compact. In [83] it has been proved that a necessary and sufficient condition to this compactness property is the following strict additivity condition

$$m(c) < m(\rho) + m(c - \rho), \quad \forall\ 0 < \rho < c. \tag{1.1.8}$$

Now we focus on the case $p \in [3, \frac{10}{3}]$. In Chapter 2, we first establish a non-existence result of minimizers of $m(c)$. This result is sharp in the sense that we explicit a threshold value of $c > 0$ that separates the existence and non-existence of the minimizers. More generally, we prove in Chapter 2 that constrained critical points of F do not exist when $c > 0$ is sufficiently small.

Before presenting precisely the non-existence results, let us indicate some properties of the function $c \to m(c)$ when $p \in [3, \frac{10}{3}]$. The study of this function is interesting for itself, but also it is a key of our approach to establish the existence or non-existence of minimizers. Let

$$c_1 := \inf\{c > 0 \ : \ m(c) < 0\}. \tag{1.1.9}$$

Theorem 1.1.2. ($\mathbb{I}$) *When $p \in (3, \frac{10}{3})$ we have*

(i) $c_1 \in (0, \infty)$;

(ii) $m(c) = 0$, *as* $c \in (0, c_1]$;

(iii) $m(c) < 0$ *and is strictly decreasing about* c*, as* $c \in (c_1, \infty)$*;*

(iv) *The function* $c \to m(c)$ *is continuous at each* $c > 0$*.*

(III) *When* $p = 3$ *or* $p = \frac{10}{3}$ *we have*

(v) *When* $p = 3$*,* $m(c) = 0$ *for all* $c > 0$*;*

(vi) *When* $p = \frac{10}{3}$*, we denote*

$$c_2 := \inf\{c > 0 \ : \ \exists\ u \in S(c) \text{ such that } F(u) \leq 0\}, \tag{1.1.10}$$

then $c_2 \in (0, \infty)$ *and*

$$\begin{cases} m(c) = 0, & \text{as } \ c \in (0, c_2); \\ m(c) = -\infty, & \text{as } \ c \in (c_2, \infty). \end{cases} \tag{1.1.11}$$

Our result concerning the existence or non-existence of minimizers of $m(c)$ is

Theorem 1.1.3. *(i) When* $p \in (3, \frac{10}{3})$*,* $m(c)$ *has a minimizer if and only if* $c \in [c_1, \infty)$*.*

(ii) When $p = 3$ *or* $p = \frac{10}{3}$*,* $m(c)$ *has no minimizer for any* $c > 0$*.*

Theorem 1.1.3 provides a fairly complete answer to the issue of global minimizers for F on $S(c)$ when $p \in [3, \frac{10}{3}]$. By contrast, when $p \in (2, 3)$, as one sees in Theorem 1.1.1, the situation is much less understood. It is only known that a minimizer exists when $c > 0$ is sufficiently small. Clearly for any $c > 0$, $m(c) < 0$ and any minimizing sequence is bounded. However in trying to develop a minimization process one faces the difficulty in ruling out the possible dichotomy of the minimizing sequences. Thus it is still an open question whether or not $m(c)$ is reached for $c > 0$ large.

To establish our non-existence results, we explicit an identity, see Lemma 2.2.1, satisfied by all the critical points of F on $S(c)$. That is, if u_0 is a critical point of F on $S(c)$, then necessarily

$$\|\nabla u_0\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_0(x)|^2 |u_0(y)|^2}{|x-y|} dxdy - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u_0|^p dx = 0. \tag{1.1.12}$$

In addition to the non-existence results of Theorem 1.1.3, we give also a more general non-existence result concerning the critical points of F on $S(c)$. Precisely

Theorem 1.1.4. *When* $p \in (3, \frac{10}{3}]$*, there exists* $\bar{c} > 0$ *such that for any* $c \in (0, \bar{c})$*, there are no critical points of* F *restricted to* $S(c)$*. When* $p = 3$*, for all* $c > 0$*,* F *does not admit critical points on the constraint* $S(c)$*.*

Theorem 1.1.4 is, up to our knowledge, the first result where a non-existence result of small L^2-norm solutions is established for (E_λ). Note however that it was independently proved by H. Kikuchi [70] and D. Ruiz [99] that when $p \in (2, 3)$ there exists a $\lambda_0 < 0$ such that (E_λ) has only trivial solution when $\lambda \in (-\infty, \lambda_0)$.

Following our study of the case $p \in [3, \frac{10}{3}]$, a systematic treatment of the case $p \in [2, \frac{10}{3}]$, as far as the existence of global minimizers is concerned, has been done recently in [34]. The main aim of [34] is to show that the previous results on this issue can be reproved by a unified approach based on the concentration-compactness principle of P. L. Lions and the use of some interpolation inequalities which generalize the Gagliardo-Nirenberg inequality (see [34] for more details). Some new results are also obtained. In [34] the authors study the existence of global minimizers of the functional

$$E_d(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy - \frac{d}{p}\int_{\mathbb{R}^3}|u|^p dx, \ d>0,$$

on the constraint $S(c)$. In the range $p \in (3, \frac{10}{3})$ they manage to give in term of some best Sobolev constants an "explicit" value for the threshold value $c_1 > 0$ (c_1 is given in (1.1.9)). Also when $p = 3$ they prove that there exists a $d_0 > 0$ such that for any $c > 0$, $m(c) < 0$ and the functional E_d has a global minimizer on $S(c)$ if $d > d_0$. On the contrary $m(c) = 0$ and E_d has no minimizer when $d < d_0$. This result which should be set in parallel with Theorem 1.1.2 ($\mathbb{III}$) (v) and Theorem 1.1.3 (ii), gives a new light on the case $p = 3$.

1.1.2 Solutions of saddle point type

In Chapter 3, we consider the case $p \in (\frac{10}{3}, 6)$. For this range of power the functional F is no more bounded from below on $S(c)$ and thus it is impossible to find a solution as a global minimizer. We shall nevertheless be able to find a critical point of F assuming $c > 0$ sufficiently small. As a first step we observe that the functional F has a mountain pass geometry on $S(c)$ for any $c > 0$.

Definition 1.1.5. Given $c > 0$, we say that F has a mountain pass geometry on $S(c)$ if there exists a $K_c > 0$, such that

$$\gamma(c) := \inf_{g\in\Gamma_c}\max_{t\in[0,1]} F(g(t)) > \max\Big\{\max_{g\in\Gamma_c} F(g(0)), \max_{g\in\Gamma_c} F(g(1))\Big\}, \tag{1.1.13}$$

holds in the set

$$\Gamma_c := \Big\{g \in C([0,1], S(c)) : \ g(0) \in A_{K_c}, F(g(1)) < 0\Big\},$$

where $A_{K_c} := \{u \in S(c) : \ \|\nabla u\|_2^2 \leq K_c\}$.

We shall look to a critical point of F on $S(c)$ at the mountain pass level $\gamma(c)$. Our result concerning the existence of solutions of (E_λ) is given by the following

Theorem 1.1.6. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. Then F has a mountain pass geometry on $S(c)$. Moreover there exists $c_0 > 0$ such that for any $c \in (0, c_0)$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}^-$ solution of (E_λ) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. In addition*

$$\lambda_c \to -\infty, \quad as \quad c \to 0. \tag{1.1.14}$$

The fact that the mountain pass geometry of F on $S(c)$ holds is based on the fact that when $p \in (\frac{10}{3}, 6)$, the nonlinearity $|u|^{p-2}u$ grows "sufficiently fast" at infinity. Having proved that F has a mountain pass geometry we know by Ekeland's variational principle (see [107]), that there exists a Palais-Smale sequence for F at the level of $\gamma(c)$. Namely a sequence $\{u_n\} \subset S(c)$ has the property as $n \to \infty$ that

$$F(u_n) \to \gamma(c), \ \text{ and } \ F'|_{S(c)}(u_n) \to 0 \ \text{ in } H^{-1}(\mathbb{R}^3).$$

To hope to obtain a critical point at the level $\gamma(c)$, one first needs to show that at least one of such sequences is bounded. However, under the assumption of Theorem 1.1.6, this aim seems to be challenging. To overcome the difficulty, we introduce the functional

$$Q(u) := \|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3}|u|^p dx, \tag{1.1.15}$$

the set

$$V(c) := \Big\{ u \in S(c) \ : \ Q(u) = 0 \Big\}$$

and we prove that

$$\gamma(c) = \inf_{u \in V(c)} F(u) > 0. \tag{1.1.16}$$

This additional variational characterization is central in our proofs. The proof of (1.1.16) is given in Lemma 3.2.6. Moreover, we show that each constrained critical point of $F(u)$ on $S(c)$ must lie in $V(c)$, namely $V(c)$ acts as a natural constraint. At this point, taking advantage of the nice "shape" of some sequence of paths $\{g_n\} \subset \Gamma_c$ such that

$$\max_{t \in [0,1]} F(g_n(t)) \to \gamma(c),$$

we construct a special Palais-Smale sequence $\{u_n\} \subset S(c)$ at the level $\gamma(c)$ which concentrates around $V(c)$, see Section 3.3 in Chapter 3. This localization leads to its boundedness but also provides the additional information that $Q(u_n) = o(1)$. This last property is crucially used in the study of the compactness of the sequence.

To obtain a critical point of F at the level $\gamma(c)$ one still needs to show that, up to some translations and passing to a subsequence, $\{u_n\}$ converges in $H^1(\mathbb{R}^3)$. Since the equation (E_λ) is set in the whole $\mathbb{R}^3$, one has to face a classical lack of compactness of the Sobolev embedding. In addition with respect to the case where one looks to a global minimizer, and in particular searches for a critical point at a strictly negative level, the fact that we deal here with strictly positive suspected critical level $\gamma(c)$ brings new difficulties. It appears hard to follow the classical vanishing-dichotomy-compactness scenario. In particular the strict subadditivity condition (1.1.8) loses its pertinence to discuss the compactness of the sequence $\{u_n\} \subset S(c)$.

To overcome this difficulty, we first study the behavior of the function $c \to \gamma(c)$. The theorem below summarizes its properties.

Theorem 1.1.7. *Let $p \in (\frac{10}{3}, 6)$, and for any $c > 0$ let $\gamma(c)$ be the mountain pass level, given by* (1.1.13)*. Then*

(i) $c \to \gamma(c)$ *is continuous at each* $c > 0$.

(ii) $c \to \gamma(c)$ *is non-increasing.*

(iii) There exists $c_0 > 0$ such that in $(0, c_0)$ the function $c \to \gamma(c)$ is strictly decreasing.

(iv) There exists $c_\infty > 0$ such that for all $c \geq c_\infty$ the function $c \to \gamma(c)$ is constant.

(v) $\lim_{c \to 0} \gamma(c) = +\infty$ *and* $\lim_{c \to \infty} \gamma(c) =: \gamma(\infty) > 0$.

We show that if the function $c \to \gamma(c)$ is strictly decreasing, then the constructed Palais-Smale sequence $\{u_n\} \subset S(c)$ converges strongly in $H^1(\mathbb{R}^3)$, up to a subsequence and translations if necessary, see Lemma 3.4.4. Thus there exists a critical point $u_c \in S(c)$ of F on $S(c)$ such that $F(u_c) = \gamma(c)$. However, as one sees in Theorem 1.1.7 (*iii*), we are only able to prove that $c \to \gamma(c)$ is strictly decreasing for $c > 0$ sufficiently small. That is the reason why our existence results in Theorem 1.1.6 are restricted for small $c > 0$. As for the other values of $c > 0$ the information that $c \to \gamma(c)$ is non-increasing permits us to reduce the problem of convergence to the one of showing that the associated Lagrange multiplier $\lambda_c \in \mathbb{R}$ is non zero (see Lemma 3.4.5). However we do prove that $\lambda_c = 0$ holds for any $c > 0$ sufficiently large (see Lemma 3.7.3). In view of this point, we conjecture that $\gamma(c)$ is not a critical value for $c > 0$ large enough. Remark 3.7.4 shows us the details in that direction.

We point out that the fact that $c \to \gamma(c)$ is non increasing could be expected in our problem. This property is in some sense the analogue of the large subadditivity condition

$$m(c) \leq m(\rho) + m(c-\rho), \quad \forall\ 0 < \rho < c. \tag{1.1.17}$$

which holds for general minimization problem [83]. To establish this property we rely on some ideas of [67] where it was done, using a so called "added mass technique" on a particular minimization problem.

To prove Theorem 1.1.7 (iv) and that $\gamma(c) \to \gamma(\infty) > 0$ as $c \to \infty$ in (v) we take advantage of some results of I. Ianni and D. Ruiz [63]. In [63] the static equation

$$-\Delta v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \ \text{ in } \ \mathbb{R}^3 \tag{1.1.18}$$

which corresponds to the case $\lambda = 0$ in (E_λ), is considered. The authors established in [63] the existence of a critical point of F in the space

$$E := \Big\{u \in D^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy < \infty\Big\}. \tag{1.1.19}$$

This critical point is a ground state solution to (1.1.18).

In addition, it is proved in [63, Theorem 6.1] that any radial solution of (1.1.18) decreases exponentially at infinity. We extend here this result to any solution of (1.1.18). More precisely we prove

Theorem 1.1.8. *Let $p \in (3,6)$ and $(u,\lambda) \in E \times \mathbb{R}$ with $\lambda \leq 0$ be a solution of (E_λ). Then there exist constants $C_1 > 0$, $C_2 > 0$ and $R > 0$ such that*

$$|u(x)| \leq C_1 |x|^{-\frac{3}{4}} e^{-C_2\sqrt{|x|}}, \ \forall\ |x| > R. \tag{1.1.20}$$

In particular, $u \in H^1(\mathbb{R}^3)$.

Theorem 1.1.8 implies that any solution of (1.1.18) belongs to $L^2(\mathbb{R}^3)$. This information is crucial to derive Theorem 1.1.7 (iv)-(v). In addition, the exponential decay property of the solutions, as we shall see later, is crucially used for the analysis of the dynamics of standing wave solutions to the Cauchy problem of (1.1.3).

The fact that $c \to \gamma(c)$ becomes constant as $c > 0$ large (which leads very likely to the conclusion that $\gamma(c)$ is not a critical level for $c > 0$ large), is due to the term $(|x|^{-1} * |u|^2)u$. In order to try to understand this, we draw a comparison between (1.1.3) and the classical nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{1.1.21}$$

In [64], L. Jeanjean considered the existence of standing waves for (1.1.21) on $S(c)$ when $p \in [\frac{10}{3}, 6)$. Then the associated functional is unbounded from below. In [64] a solution was obtained for any given $c > 0$ after having shown that the associated Lagrange multiplier is strictly negative for any $c > 0$. In this thesis we complement and enlighten this result by showing that the mountain pass value $\tilde{\gamma}(c)$ associated with (1.1.21) is strictly decreasing as a function of $c > 0$ and that $\tilde{\gamma}(c) \to 0$ as $c \to \infty$.

The fact that (1.1.16) holds and that any constrained critical point of F lies in $V(c)$ implies that the solutions found in Theorem 1.1.6 can be considered as ground states within the solutions having the same L^2-norm.

Now we denote the set of minimizers of F on $V(c)$ as

$$\mathcal{M}_c := \{u_c \in V(c) \ : \ F(u_c) = \inf_{u \in V(c)} F(u)\}, \tag{1.1.22}$$

where we have enlarged the space of functions to $H^1(\mathbb{R}^3, \mathbb{C})$. This extension will be required to discuss later the dynamics of the standing waves and in particular their orbital stability.

Theorem 1.1.9. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$, there exists a $\lambda_c \leq 0$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves (E_λ).*

Clearly to prove Theorem 1.1.9, one needs to show that any minimizer of $F(u)$ on $V(c)$ is a critical point of F restricted to $S(c)$. As additional properties of elements of $\mathcal{M}_c$ we obtain:

Lemma 1.1.10. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$ be arbitrary. Then*

(i) If $u_c \in \mathcal{M}_c$ then also $|u_c| \in \mathcal{M}_c$.

(ii) Any minimizer $u_c \in \mathcal{M}_c$ has the form $e^{i\theta}|u_c|$ for some $\theta \in \mathbb{S}^1$ and $|u_c(x)| > 0$ a.e. in $\mathbb{R}^3$.

In view of Lemma 1.1.10, each elements of $\mathcal{M}_c$ is a real positive function multiplied by a constant complex factor. This lemma shows in particular that least energy solutions can be searched for only within real valued functions.

Remark 1.1.11. A natural question that arises, as a consequence of Theorem 1.1.9, is why not search for solutions of (E_λ) with a prescribed norm by directly minimizing F on $V(c)$. However starting from an arbitrary minimizing sequence $\{u_n\} \subset V(c)$ and trying to show its convergence seem to be challenging. From the definition of $V(c)$ it is easy to prove that any minimizing sequence is bounded in $H^1(\mathbb{R}^3)$ and thus we can assume that $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ for some $\bar{u} \in H^1(\mathbb{R}^3)$. Also ruling out the vanishing is not a problem as it can be seen from Lemma 3.4.2. But to show that the dichotomy does not occur it seems necessary to know that $\bar{u} \in V(||\bar{u}||_2^2)$. For our Palais-Smale sequence we use, in Lemma 3.4.4, the information that $\bar{u} \in H^1(\mathbb{R}^3)$ is a non-trivial solution of (E_λ). Then by Lemma 3.4.3, $Q(\bar{u}) = 0$ and $\bar{u} \in V(||\bar{u}||_2^2)$. For an arbitrary minimizing sequence it does not seem possible to show that the weak limit $\bar{u} \in H^1(\mathbb{R}^3)$ belongs to $V(||\bar{u}||_2^2)$. Having such information seems to require some information on the derivative of F along the sequence and that is why we introduce Palais-Smale sequences to solve our minimization problem.

1.1.3 Dynamics of standing waves

Now we turn our attention to the analysis of the dynamics of standing waves for (1.1.3). First recall from [37] that, the Cauchy problem of (1.1.3) is locally well posed in $H^1(\mathbb{R}^3, \mathbb{C})$ and keeps the quantities of energy and charge conserved in time. Namely,

Proposition 1.1.12. *Assume that $p \in (2, 6)$. Then for every $\psi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$, there exists a $T = T(\|\psi_0\|) > 0$ and a unique solution $\psi(t) \in C([0, T), H^1(\mathbb{R}^3, \mathbb{C}))$ of (1.1.3) with the* *initial datum* $\psi(0) = \psi_0$, *such that either* $T = \infty$ *(we say that* $\psi(t)$ *exists globally) or* $\lim_{t \to T} \|\nabla\psi(t)\|_2 = \infty$ *(we say that* $\psi(t)$ *blows up in a finite time), and satisfy that*

$$F(\psi(t)) = F(\psi_0), \ \|\psi(t)\|_2 = \|\psi_0\|_2, \quad \forall \ t \in [0, T).$$

In addition, if $\psi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ *such that* $|x|\psi_0 \in L^2(\mathbb{R}^3, \mathbb{C})$, *then the virial identity*

$$\frac{d^2}{dt^2}\|x\psi(t)\|_2^2 = 8Q(\psi(t)),$$

holds for all $t \in [0, T)$, *where* $Q(\psi)$ *is given by* (1.1.15).

We say that standing waves are orbitally stable or unstable in the following sense:

Definition 1.1.13. (i) Let $\Omega \subset H^1(\mathbb{R}^3, \mathbb{C})$. Then the set Ω is orbitally stable, if for any $\varepsilon > 0$, there exists a $\delta > 0$ with the following property: for each $\psi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfying $\inf_{\phi \in \Omega} \|\psi_0 - \phi\| < \delta$, there holds that

$$\sup_{t>0} \inf_{\phi \in \Omega} \|\psi(t) - \phi\| < \varepsilon,$$

where $\psi(t)$ is the solution of (1.1.3) with the initial datum $\psi(0) = \psi_0$. Otherwise, we say that Ω is orbitally unstable.

(ii) A standing wave $e^{-i\lambda t}u(x)$ of (1.1.3) is said to be orbitally stable if the orbit $\{e^{i\theta}u(\cdot - y) : \ \forall \ \theta \in \mathbb{R}, \forall \ y \in \mathbb{R}^3\}$ is stable.

(iii) A standing wave $e^{-i\lambda t}u(x)$ of (1.1.3) is said to be strongly unstable in the sense that for any $\varepsilon > 0$, there exists a $\psi_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfying $\|\psi_0 - u(x)\| < \varepsilon$, such that the solution $\psi(t)$ of (1.1.3) with $\psi(0) = \psi_0$ blows up in a finite time.

Following the approach of Cazenave-Lions [36], it was proved in [14, 15, 105] that

Theorem 1.1.14. *Assume that* $p \in (2, 3) \cup (3, \frac{10}{3})$. *Let* $u_c \in S(c)$ *be a minimizer of* $m(c)$, *obtained in Theorem 1.1.1. Then the orbit*

$$\{e^{i\theta}u_c(\cdot - y) : \ \forall \ \theta \in \mathbb{R}, \forall \ y \in \mathbb{R}^3\}$$

is stable. Namely the standing wave $e^{-i\lambda_c t}u_c(x)$ *is orbitally stable, where* $\lambda_c \in \mathbb{R}$ *is the corresponding Lagrange parameter of* u_c.

The key to prove Theorem 1.1.14 is the information that all minimizing sequences of $m(c)$ are pre-compact. This point has been established by [14, 15, 102, 105] in the proof of Theorem 1.1.1.

In the case $p \in (\frac{10}{3}, 6)$, the global existence in time of solutions of the Cauchy problem associated with (1.1.3) does not hold for arbitrary initial condition, see e.g. [72]. However we are able to prove the following global existence result.

Theorem 1.1.15. *Let* $p \in (\frac{10}{3}, 6)$ *and* $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ *be an initial condition associated with* (1.1.3) *with* $c = ||u_0||_2^2$. *If*

$$Q(u_0) > 0 \text{ and } F(u_0) < \gamma(c),$$

then the solution of (1.1.3) *with the initial condition* u_0 *exists globally in time.*

In Remark 3.8.2 we prove that the set

$$\mathcal{O} = \{u_0 \in S(c) \ : \ Q(u_0) > 0 \text{ and } F(u_0) < \gamma(c)\}$$

is not empty.

Theorem 1.1.15 is, at its modest level, in the spirit of recent works [48, 60, 93, 94, 109] which try to understand deeply the dynamics of some nonlinear equations.

In what follows, we prove that the standing waves corresponding to elements of $\mathcal{M}_c$ are strongly unstable.

Theorem 1.1.16. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$, the standing wave $e^{-i\lambda_c t}u_c$ of* (1.1.3) *where $\lambda_c \in \mathbb{R}$ is the Lagrange multiplier, is strongly unstable.*

In view of (1.1.16), this theorem yields the strong instability of the standing waves we obtained in Theorem 1.1.6. The proof of Theorem 1.1.16 borrows elements of the original approach of H. Berestycki and T. Cazenave [18]. The starting point is the variational characterization of $u_c \in \mathcal{M}_c$ and the decay estimates, established in Theorem 1.1.8, prove crucial to use the virial identity.

For previous results concerning the instability of standing waves of (1.1.3), we refer to H. Kikuchi [72] (see also [71]). In [72], working in the subspace of radially symmetric functions, it is proved that for $\lambda < 0$ fixed and $p \in (\frac{10}{3}, 6)$, the equation (1.1.3) admits a ground state which is strongly unstable.

As a consequence of Theorem 1.1.16, we obtain

Theorem 1.1.17. *Let $p \in (\frac{10}{3}, 6)$. Any ground state of* (1.1.18) *is strongly unstable.*

An equation like (1.1.18) where $\lambda = 0$, is usually referred to as of *zero mass* type. Actually, in the *zero mass* case, there seems to be few results of stability or instability of standing waves. We are only aware of a stability result of M. Kaminaga and M. Ohta [68].

1.1.4 Multiplicity of normalized solutions

In Chapter 4, we establish the existence of infinitely many normalized solutions for equation (E_λ). Precisely, we prove

Theorem 1.1.18. *Assume that $p \in (\frac{10}{3}, 6)$. There exists a $c_0 > 0$ such that for any $c \in (0, c_0)$, the equation (E_λ) admits an unbounded sequence of distinct pairs of solutions $(\pm u_n, \lambda_n)$ with $\|u_n\|_2^2 = c$ and $\lambda_n < 0$ for each $n \in \mathbb{N}$.*

Clearly the sequence of solutions $(\pm u_n, \lambda_n) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ will be obtained as critical points and associated Lagrange multipliers of the functional F on the L^2-norm constraint $S(c)$.

In view that F is unbounded from below on $S(c)$ when $p \in (\frac{10}{3}, 6)$, the genus of the sublevel set $F^\alpha := \{u \in S(c) : F(u) \leq \alpha\}$ is always infinite. Thus to obtain the existence of infinitely many solutions, classical arguments based on the Kranoselski genus, see [107], do not apply.

Since we are not concerned here, as it was the case in Chapters 2 and 3, by the search of least energy solutions, we can work in the subspace $H_r^1(\mathbb{R}^3)$ of radially symmetric functions.

It is classical that a critical point of F restricted to $H^1_r(\mathbb{R}^3) \cap S(c)$ is a critical point of F restricted to $H^1(\mathbb{R}^3) \cap S(c)$. The advantage of working in $H^1_r(\mathbb{R}^3)$ is that the embedding of $H^1_r(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ is compact for $q \in (2, 6)$. However, as it can easily be checked, despite this property, F restricted to $S(c)$ does not satisfy the Palais-Smale condition.

To overcome these difficulties we rely on a recent work of T. Bartsch and S. De Valeriola [12]. In [12] the authors consider the problem of finding infinitely many critical points for

$$\mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx, \tag{1.1.23}$$

on the constraint

$$S_r(c) := \Big\{u \in H^1_r(\mathbb{R}^3) : \ \|u\|_2^2 = c, c > 0\Big\}, \tag{1.1.24}$$

when $p \in (\frac{10}{3}, 6)$. Actually in [12] more general nonlinearities can be handled and in any dimension $N \geq 2$.

In the problem treated in [12] the difficulties presented above already exist. To overcome these difficulties the authors present a new type of linking geometry for the functional $\mathcal{E}$ on $S_r(c)$. This geometry is, according to the authors of [12], motivated by the fountain theorem (see [11]). In [12] to set up a min-max scheme and identify a sequence $\{l_n\} \subset \mathbb{R}$, $l_n \to \infty$ of suspected critical levels, the cohomological index for spaces with an action on the group $G = \{-1, 1\}$ is used. Indeed observe that the functional $\mathcal{E}$ is even, this is also the case of F. This index which was introduced in [31] permits to establish the key intersection property, see [12, Lemma 2.3] or our Lemma 4.2.3. The fact that the suspected critical levels l_n do correspond to critical levels is then obtained using ideas from [64]. The key point is the construction, for each fixed $n \in \mathbb{N}$, of a bounded Palais-Smale sequence associated with l_n. In that aim one introduces an auxiliary functional which permits to incorporate into the variational procedure the information that any critical point of $\mathcal{E}$ on $S_r(c)$ must satisfy a version of Pohozaev identity. Having obtained the boundedness of a Palais-Smale sequence it remains to show that it converges. The information that the associated Lagrange multiplier is strictly negative is here crucially used.

In our proof of Theorem 1.1.18 we follow closely the strategy of [12]. The restriction that $c \in (0, c_0)$ originates in the need to show that the suspected associated Lagrange multipliers are strictly negative. This property is used to show that the weak limit of our Palais-Smale sequences do belong to $S_r(c)$. A similar limitation on $c > 0$ was already necessary for the existence of just one critical point. More generally Chapter 4 makes a strong use of results derived in Chapter 3.

Up to our knowledge, Theorem 1.1.18 is the first result in the literature on the existence of infinitely many L^2-normalized solutions for equation (E_λ). Previous results had already been obtained when $\lambda \in \mathbb{R}$ is a fixed parameter. We refer to [4, 8, 42, 104] and their references in that direction.

1.2 Quasi-linear Schrödinger Equations

In Chapters 5 and 6, we deal with a class of quasi-linear Schrödinger equations of the form

$$i\partial_t\varphi + \Delta\varphi + \varphi\Delta|\varphi|^2 + |\varphi|^{p-1}\varphi = 0, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.2.1}$$

where $N \in \mathbb{N}$ and the unknown $\varphi = \varphi(t,x) : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$ is a complex valued function. The equation (1.2.1) is viewed as a special case of the following general quasi-linear Schrödinger equations

$$i\partial_t\varphi + \Delta\varphi + f(|\varphi|^2)\varphi - \sigma\Delta h(|\varphi|^2)h'(|\varphi|^2)\varphi, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.2.2}$$

where $\sigma \in \mathbb{R}$ is a constant, $V(x)$ is a given potential function, and f, h are real functions. Equation (1.2.2), has been derived as models of several physical phenomena corresponding to various types of functions h. For example, when $h(s) = s^{1/2}$ the equation (1.2.2), is called the superfluid film equation, was used in plasma physics by S. Kurihara [74]. When $h(s) = (1+s)^{1/2}$, (1.2.2) models the self-channeling of high-power ultra short laser in matter [22, 46, 98]. Moreover, depending on different types of f and h, (1.2.2) also appears in other models of plasma physics and fluid mechanics [55, 85], in the theory of Heisenberg ferromagnets and magnons [13], in dissipative quantum mechanics [58], and in condensed matter theory [91].

By letting $h(s) = s^{1/2}, \sigma = 1$ and $f(s) = s^{(p-1)/2}, p > 1$, (1.2.2) is reduced to (1.2.1). In [26, 27, 28, 29, 57], (1.2.1) has been introduced to study a model of self-trapped electrons in quadratic or hexagonal lattices.

From the physical as well as mathematical point of view, a central issue is the existence and dynamics of standing waves of (1.2.1). Observe that the standing wave $e^{-i\lambda t}u(x)$ where $\lambda \in \mathbb{R}$, solves (1.2.1) if and only if $u(x)$ satisfies the following stationary equation

$$-\Delta u - u\Delta(u^2) - \lambda u - |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^N. \tag{P_λ}$$

A function u is called a weak solution of equation (P_λ) if

$$Re \int_{\mathbb{R}^N} \Big(\nabla u \cdot \nabla\phi + \nabla(|u|^2) \cdot \nabla(u\phi) - \lambda u\phi - |u|^{p-1}u\phi\Big)\, dx = 0 \tag{1.2.3}$$

for all $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$.

In (P_λ), when $\lambda \in \mathbb{R}$ is a fixed parameter, the existence and multiplicity of solutions of (P_λ) have been intensively studied during the last decade. See [5, 40, 41, 50, 86, 87, 88, 89, 96, 97, 101] and their references therein. We also refer to the works [1, 5, 54, 103] for the uniqueness of ground states of (P_λ). Ground states here mean solutions of (P_λ) which minimize among all nontrivial solutions of (P_λ) the associated energy functional

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx,$$

defined on the natural space

$$\mathcal{X} := \Big\{u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx < \infty\Big\}.$$

It is easy to check that u is a weak solution of (P_λ) if and only if

$$I'_\lambda(u)\phi := \lim_{t\to 0^+} \frac{I_\lambda(u+t\phi) - I_\lambda(u)}{t} = 0,$$

for every direction $\phi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$.

Compared with semi-linear equations such as (E_λ), the search of solutions of (P_λ) presents a major new difficulty. The functional associated with the quasi-linear term $u\Delta(|u|^2)$,

$$V(u) := \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx,$$

is non differentiable in the space $\mathcal{X}$ as soon as $N \geq 2$. When $N = 1$ it is of class C^1 because of the inclusion $W^{1,2}(\mathbb{R}) \subset L^\infty(\mathbb{R})$. To overcome this difficulty, various arguments have been developed. First in [87, 97], solutions of (P_λ) are obtained by minimizing the functional I_λ on the set

$$\Big\{u \in \mathcal{X} : \int_{\mathbb{R}^N} |u|^{p+1} dx = 1\Big\}.$$

In the proofs of [87, 97] the non-differentiability of I_λ essentially does not come into play. Alternatively in [40, 88], by a change of unknown, the quasi-linear problem (P_λ) is transformed into a semi-linear problem. For that semi-linear problem standard variational methods can be applied to yield a solution. Also in [89, 101] the authors have developed an approach based on the use of a Nehari manifold by which one reduces the search of solutions of (P_λ) to the problem of showing that the functional I_λ has a global minimizer on the Nehari manifold. Since these pioneering works there has been a large literature on the equation (P_λ) where are addressed the questions of multiplicity, of concentration type issue or of critical exponent type.

In this thesis, we focus on the existence of solutions of (P_λ) having a prescribed L^2-norm. Up to our knowledge the first results in that direction were obtained in [41], see also [38]. In [41], for any given $c > 0$, the authors consider the minimization problem

$$\bar{m}(c) := \inf_{u \in \bar{S}(c)} J(u), \tag{1.2.4}$$

where

$$\bar{S}(c) := \Big\{u \in \mathcal{X} : \|u\|_2^2 = c\Big\}. \tag{1.2.5}$$

Here the functional $J : \bar{S}(c) \to \mathbb{R}$, is defined as

$$J(u) := \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx, \tag{1.2.6}$$

with $p \in (1, 2 \cdot 2^* - 1)$. In [41, Theorem 4.6]), it is shown that for each minimizer $u \in \bar{S}(c)$ of $\bar{m}(c)$, there exists a Lagrange parameter $\lambda < 0$ such that the couple (u, λ) solves (P_λ).

It is also shown in [41] that if $p < 3 + \frac{4}{N}$ then $\bar{m}(c) > -\infty$ for any $c > 0$. On the contrary, when $p > 3 + \frac{4}{N}$, we have $\bar{m}(c) = -\infty$ for any $c > 0$.

Remark 1.2.1. The key point to show that $\bar{m}(c) > -\infty$ if $p \in (1, 3 + \frac{4}{N})$ is the use of the following Gagliardo-Nirenberg inequality which was proved in [41, (4.5)]. That is, for some $K > 0$ depending only on N and p, there holds for any $u \in \mathcal{X}$ that,

$$\int_{\mathbb{R}^N} |u|^{p+1} dx \leq K\Big(\int_{\mathbb{R}^N} |u|^2 dx\Big)^{1-\theta}\Big(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx\Big)^{\frac{\theta N}{N-2}}, \tag{1.2.7}$$

where $\theta = \frac{(p-1)(N-2)}{2(N+2)}$. One notes that $\frac{\theta N}{N-2} < 1$ when $p < 3 + \frac{4}{N}$, then the negative term in J can be controlled by the second one, which leads to $\bar{m}(c) \neq -\infty$. Recall that the corresponding functional setting associated with

$$-\Delta u - \lambda u = |u|^{p-1}u, \quad \text{in } \mathbb{R}^N, \tag{1.2.8}$$

is given, on $H^1(\mathbb{R}^N)$, by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx$$

and

$$d(c) := \inf\{I(u) : \ u \in H^1(\mathbb{R}^N), \ \|u\|_2^2 = c\}.$$

In this case to control the negative term in I so that $d(c) > -\infty$, one needs to require that $p < 1 + \frac{4}{N}$. These considerations show that the exponent $3 + \frac{4}{N}$ for (P_λ) plays the same role as $1 + \frac{4}{N}$ for (1.2.8). In addition, the inequality (1.2.7), and the definition of $\mathcal{X}$, permit to extend the range of the power for the negative term to $1 < p < 2 \cdot 2^* - 1$. Indeed, the exponent $p = 2 \cdot 2^* - 1$ is critical with respect to the existence of solutions for (P_λ), see [89, 101] for instance.

It is also proved in [41] that when $p \in (1, 1 + \frac{4}{N})$ a minimizer of $\bar{m}(c)$ exists for all $c > 0$. When $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N}]$ it is claimed that there exists a $c(p, N) > 0$ such that minimizers of $\bar{m}(c)$ do not exist for $c < c(p, N)$ but do exist for $c > c(p, N)$. However there are some gaps in the proofs of [41].

In Chapter 5, we focus on the range $p \in [1+\frac{4}{N}, 3+\frac{4}{N}]$, and we try to clarify and extend the results of [41]. In particular we settle the question of existence for the threshold value $c(p, N)$. Our main result is as follows.

Theorem 1.2.2. *Assume that $p \in (1, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (1, \infty)$ if $N = 1, 2$. Then*

(1) Concerning the properties of the function $c \to \bar{m}(c)$, we have

i) For all $c > 0$, $\bar{m}(c) \in (-\infty, 0]$ as $p \in (1, 3 + \frac{4}{N})$;

ii) For all $c > 0$, $\bar{m}(c) = -\infty$ as $p \in (3 + \frac{4}{N}, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (3 + \frac{4}{N}, \infty)$ if $N = 1, 2$;

iii) For $p = 3 + \frac{4}{N}$, there exists a $c_N > 0$, given by

$$c_N := \inf\{c > 0 : \ \exists\, u \in \bar{S}(c) \text{ such that } J(u) \leq 0\},$$

such that

$$\begin{cases} \bar{m}(c) = 0, & \text{as } c \in (0, \ c_N); \\ \bar{m}(c) = -\infty, & \text{as } \ c \in (c_N, \infty). \end{cases}$$

(2) ([41, Theorems 1.12]) When $p \in (1, 1 + \frac{4}{N})$, for all $c > 0$, $\bar{m}(c) < 0$ and $\bar{m}(c)$ has a minimizer.

(3) When $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$, there exists a $c(p, N) > 0$, given by

$$c(p, N) := \inf\{c > 0 : \ \bar{m}(c) < 0\}, \tag{1.2.9}$$

such that

i) If $c \in (0, c(p, N))$, $\bar{m}(c) = 0$ and $\bar{m}(c)$ has no minimizer;

ii) If $c = c(p, N)$, $\bar{m}(c) = 0$ and $\bar{m}(c)$ admits a minimizer;

iii) If $c \in (c(p, N), \infty)$, $\bar{m}(c) < 0$ and $\bar{m}(c)$ admits a minimizer.

(4) When $p = 3 + \frac{4}{N}$, for all $c > 0$, $\bar{m}(c)$ admits no minimizers.

(5) *([41, Theorems 1.9]) The standing waves obtained as minimizers of $\bar{m}(c)$ are orbitally stable.*

Also we prove an analogue of Theorem 1.1.4 concerning the non-existence of solutions with small L^2-norm.

Theorem 1.2.3. *Assume that $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N}]$ holds. Then there exists a $\hat{c} > 0$ such that for all $c \in (0, \hat{c})$, the equation (P_λ) has no solution belonging to $\bar{S}(c)$.*

Key to our non existence result is an identity similar to (1.1.12). Actually if $v_0 \in \bar{S}(c)$ is a solution of (P_λ), it must satisfy the identity

$$\|\nabla v_0\|_2^2 + (N+2)\int_{\mathbb{R}^N} |v_0|^2|\nabla v_0|^2 dx - \frac{N(p-1)}{2(p+1)}\int_{\mathbb{R}^N} |v_0|^{p+1} = 0. \tag{1.2.10}$$

This identity combined with the assumption that $||v_0||_2^2 = c$ permits to show that (P_λ) has no solution with small L^2-norm.

In Chapter 6, we consider the range $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$, $N \geq 1$. From Theorem 1.2.2 (3), we know that in this range the functional J has, for $c \geq c(p, N)$, a critical point on $\bar{S}(c)$, which is a global minimizer. Here we extend this result in two directions. First we prove that there exists a $c_0 \in (0, c(p, N))$ such that, for each $c \in (c_0, c(p, N))$ the functional J admits on $\bar{S}(c)$ a local minimizer. By Theorem 1.2.2 (3) i), this local minimizer can not be a global one. Secondly we show that when $c \in (c_0, \infty)$ the functional J admits on $\bar{S}(c)$ a second critical point. This critical point is of mountain pass type. Note that since J is not differentiable we must give a meaning to what we call a critical point of J on $\bar{S}(c)$. By definition it will be a solution of (P_λ) belonging to $\bar{S}(c)$.

Theorem 1.2.4. *Assume that $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ if $N = 1, 2, 3$ and $p \in (1 + \frac{4}{N}, \frac{N+2}{N-2}]$ if $N \geq 4$. Then there exists a $c_0 \in (0, c(p, N))$ such that for any $c \in (c_0, \infty)$ the functional J admits two critical points u_c and v_c on $\bar{S}(c)$. In addition*

(1) *$J(u_c) > J(v_c)$ for any $c \in (c_0, \infty)$.*

(2) *$J(u_c) > 0$ for all $c \in (c_0, \infty)$ and $J(u_c)$ is a mountain pass level.*

(3) $$J(v_c) \begin{cases} > 0, & \textit{if } \; c \in (c_0, c(p, N)); \\ = 0, & \textit{if } \; c = c(p, N); \\ < 0, & \textit{if } \; c \in (c(p, N), \infty). \end{cases}$$

Also v_c is a local minimum of J when $c \in (c_0, c(p, N))$ and a global minimum of J when $c \in [c(p, N), \infty)$.

(4) *u_c and v_c are Schwarz symmetric functions.*

(5) *There exist Lagrange multipliers $\lambda_c < 0$ and $\beta_c < 0$ such that (u_c, λ_c) and (v_c, β_c) solve (P_λ).*

Remark 1.2.5. In the case $c \in [c(p, N), \infty)$ the critical point v_c is just a global minimizer already obtained in [41, 65] whose existence is recalled in Theorem 1.2.2 (3).

From Theorem 1.2.3, we know that no critical points of J on $\bar{S}(c)$ exist when $c \in (0, \hat{c})$ for some $\hat{c} > 0$ small enough. But it is still an open question whether or not we can take $c_0 = \hat{c}$ in Theorem 1.2.4. Already it would be interesting to know if the set of

$c \in (0, c(p, N)]$ for which one can find the two critical points u_c and v_c of Theorem 1.2.4 is an interval.

The following figure gives us an intuition of the above results concerning the existence of critical points of J on $\bar{S}(c)$ when $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$.

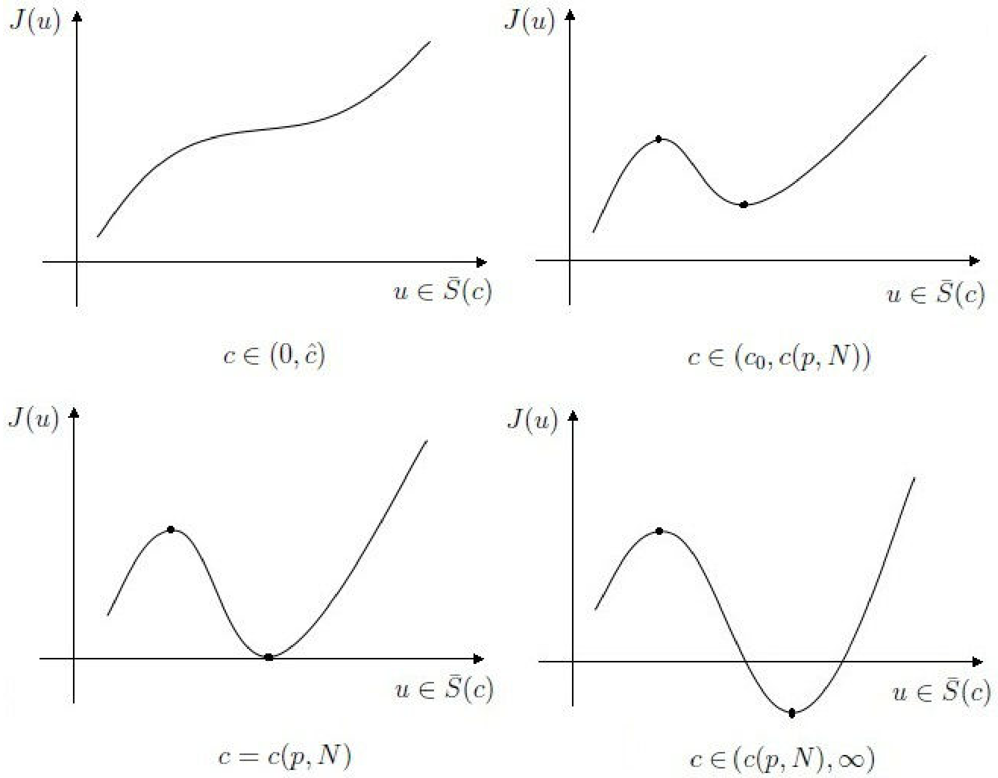

Figure 1.1

To overcome the lack of differentiability of J, we apply a perturbation method recently developed in [86]. That is, we consider first the perturbed functional

$$J_\mu(u) := \frac{\mu}{4} \int_{\mathbb{R}^N} |\nabla u|^4 dx + J(u), \tag{1.2.11}$$

where $\mu \in (0, 1]$ is a parameter. For any given $c > 0$, we denote

$$\Sigma_c := \Big\{ u \in W^{1,4} \cap W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c \Big\}.$$

One may observe that J_μ is well-defined and C^1 in Σ_c (see [86]).

The idea is to look to critical points of J_μ, for each $\mu > 0$ small and then, having obtained these critical points, to show that they converge to suitable critical points of J.

A first critical point u_μ^c of J_μ is obtained at a critical value $\gamma_\mu(c) > 0$ which corresponds to a mountain pass level. When $c \in (c_0, c(p, N))$ a second critical point v_μ^c is obtained as a local minimizer of J_μ. The corresponding energy level $\tilde{m}_\mu(c)$ is strictly positive. To derive these results we first check the geometric properties of J_μ allowing to search for such critical points. To show that these critical levels are actually reached, several difficulties have to be overcome. Since J_μ is coercive on Σ_c any Palais-Smale sequence $\{u_n\} \subset \Sigma_c$ is bounded and thus we can assume that $u_n \rightharpoonup u_c$. It is also standard to show that there exists a $\lambda_c \in \mathbb{R}$ such that $J'_\mu(u_c) - \lambda_c u_c = 0$. Finally we mention that, by constructing Palais-Smale sequences which consist of almost Schwarz symmetric functions we can avoid any problems related to possible dichotomy of our sequences, in the sense of P. L. Lions [83]. The first main difficulty is to show that $u_c \neq 0$. To overcome it we need, for both $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$, to establish the existence of Palais-Smale sequences having the additional property that $Q_\mu(u_n) \to 0$. Here $Q_\mu(u) = 0$ corresponds to the identity (1.2.10).

In the case of $\gamma_\mu(c)$ the existence of such Palais-Smale sequence is proved using the trick, first introduced in [64], to construct an auxiliary functional on $\Sigma_c \times \mathbb{R}$. This trick, which has been used recently on various problems [8, 59, 95] permits to incorporate in the variational procedure the information that any critical point of J_μ on $\Sigma(c)$ must satisfy $Q_\mu(u) = 0$. For $\tilde{m}_\mu(c)$ we can *directly* construct a minimizing sequence $\{u_n\} \subset \Sigma(c)$ satisfying $Q_\mu(u_n) = 0, \forall n \in \mathbb{N}$. It readily leads to the fact that the weak limit of the associated Palais-Smale sequence is non trivial.

Another main difficulty is to show that the weak limit u_c does belong to Σ_c, namely that $||u_c||_2^2 = c$. For this we need to require that $\lambda_c \in \mathbb{R}$ satisfies $\lambda_c < 0$. Here, and only here, comes the need to restrict our result from the *natural* range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$ for any $N \geq 1$ to the range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$ when $N = 1, 2, 3$ and $(1 + \frac{4}{N}, \frac{N+2}{N-2})$ when $N \geq 4$. It is not clear to us if it is possible to prove that $\lambda_c < 0$ for our critical points in all the range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$. Also we do not know if $\lambda_c < 0$ is necessary to get a critical point on Σ_c. However let us mention that in [16] we faced a similar issue but there strong indications incline to think it is necessary for the suspected Lagrange multipliers to be strictly negative.

Having proved the existence of the critical points u_μ^c and v_μ^c at the levels $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$ respectively we pass to the limit $\mu \to 0$ and we show that $u_\mu^c \to u_c$ and $v_\mu(c) \to v_c$ where u_c and v_c are as presented in Theorem 1.2.4. In this part we strongly rely on the machinery developed in [86].

This chapter also contains a result on the behavior of the Lagrange multipliers corresponding to the global minimizers of J.

Lemma 1.2.6. *Assume that* $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ *and that* $c \in [c(p, N), \infty)$. *Let* v_c *be a global minimizer of* J *on* $\bar{S}(c)$ *and* $\beta_c < 0$ *be its Lagrange multiplier. Then*

$$\beta_c \to -\infty, \quad \textit{as} \quad c \to \infty.$$

Finally, we present a relationship between the ground states of (P_λ) and the global minimizers of $\bar{m}(c)$.

Theorem 1.2.7. *Assume that* $p \in (1, 3 + \frac{4}{N})$. *For some* $c > 0$ *let* $u_c \in \bar{S}(c)$ *be a global minimizer of* $m(c)$ *and* $\beta_c < 0$ *be its Lagrange multiplier. Then* u_c *is a ground state solution of* (P_λ) *with* $\lambda = \beta_c$.

Note however that the converse of Theorem 1.2.7 does not hold in general. Indeed on one hand our mountain pass solution is non negative. On the other hand we know in several cases that (P_λ) has a unique non negative solution when $\lambda > 0$ is fixed. This is the case in particular when $N = 1$, see [41, Theorem 1.3] or [5]. Thus when this uniqueness property holds our mountain pass solution must necessarily be a ground state. This observation shows that not all ground states of (P_λ) for $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ can be obtained as minimizers of J on the corresponding constraint.

Remark 1.2.8. From Theorem 1.2.7 we deduce that any global minimizer u_c has a given sign and that $|u_c|$ is a radially symmetric, decreasing function with respect to one point. This follows directly from [41, Theorem 1.3].

Finally, we end this thesis by giving in Chapter 7 some remarks of our works and also some perspectives on the problems we studied.

Chapter 2

Sharp non-existence results of prescribed L^2-norm solutions for a class of Schrödinger-Poisson-Slater equations

2.1 Introduction

The following stationary nonlinear Schrödinger-Poisson equation

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0 \ \text{ in } \ \mathbb{R}^3, \tag{E_λ}$$

where $p \in (2, 6)$ and $\lambda \in \mathbb{R}$ has attracted considerable attention in the recent period. Part of the interest is due to the fact that a pair $(u(x), \lambda)$ solution of (E_λ) corresponds a standing wave $\phi(x) = e^{-i\lambda t}u(x)$ of the evolution equation

$$i\partial_t \phi + \Delta \phi - (|x|^{-1} * |\phi|^2)\phi + |\phi|^{p-2}\phi = 0 \ \text{ in } \ \mathbb{R}^+ \times \mathbb{R}^3. \tag{2.1.1}$$

This class of Schrödinger type equations with a repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree-Fock equation describing a quantum mechanical system of many particles, see for instance [10, 81, 84, 92]. For physical reasons, solutions are searched in $H^1(\mathbb{R}^3)$.

In this chapter, we are concerned with solutions of (E_λ), which is in $H^1(\mathbb{R}^3)$ and have a prescribed L^2-norm. More precisely, for given $c > 0$ we look to

$$(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R} \ \text{ solution of } \ (E_\lambda) \text{ with } \|u_c\|_2^2 = c.$$

To this purpose, solutions of (E_λ) are considered as constrained critical points of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p\, dx$$

on the constraint

$$S(c) := \Big\{ u \in H^1(\mathbb{R}^3) : \ \|u\|_2^2 = c, \ c > 0 \Big\}.$$

The parameter $\lambda \in \mathbb{R}$, in this situation, can't be fixed any longer and it is determined by the corresponding Lagrange multiplier.

It is well known, see e.g. [99], that for any $p \in (2,6)$, F is a well defined and C^1-functional. We set

$$m(c) := \inf_{u \in S(c)} F(u).$$

Clearly minimizers of $m(c)$ are critical points of F restricted to $S(c)$, and thus solutions of (E_λ). Also it can be checked in many cases that the set of minimizers is orbitally stable under the flow of (2.1.1). Thus the search of minimizers can provide us some information on the dynamics of (2.1.1).

By scaling arguments, see Remark 2.1.3, it is readily seen that for any $c \in (0,\infty)$, $m(c) \in (-\infty, 0]$ if $p \in (2, \frac{10}{3})$ and $m(c) = -\infty$ if $p \in (\frac{10}{3}, 6)$. When $m(c) > -\infty$, the existence of minimizers of $m(c)$ has been studied in [14, 15, 102], see also [34, 72] for a closely related problem. In [102], the authors prove the existence of minimizers when $p = \frac{8}{3}$ and $c \in (0, c_0)$ for a suitable $c_0 > 0$. It is shown in [15] that a minimizer exists if $p \in (2,3)$ and $c > 0$ is small enough, and in [14] that when $p \in (3, \frac{10}{3})$, $m(c)$ admits a minimizer for any $c > 0$ sufficiently large.

The aim of this chapter is to establish non-existence results of minimizers for $m(c)$ and more generally of constrained critical points of F on $S(c)$ in the range $p \in [3, \frac{10}{3}]$. As we shall see, our results are sharp in the sense that we explicit a threshold value of $c > 0$ separating existence and non-existence of minimizers.

Before to proceed, we first give a detailed study of the function $c \to m(c)$ when $p \in [3, \frac{10}{3}]$. This study is, we believe, interesting for itself, but it is also a key to establish the existence or the non-existence of minimizers. Let

$$c_1 := \inf\{c > 0 \ : \ m(c) < 0\}. \tag{2.1.2}$$

Theorem 2.1.1. ($\mathbb{I}$) *When $p \in (3, \frac{10}{3})$ we have*

(i) $c_1 \in (0, \infty)$*;*

(ii) $m(c) = 0$*, as* $c \in (0, c_1]$*;*

(iii) $m(c) < 0$ *and is strictly decreasing about* c*, as* $c \in (c_1, \infty)$*.*

($\mathbb{II}$) *When $p = 3$ or $p = \frac{10}{3}$ we have*

(iv) When $p = 3$*,* $m(c) = 0$ *for all* $c > 0$*;*

(v) When $p = \frac{10}{3}$*, we denote*

$$c_2 := \inf\{c > 0 \ : \ \exists\ u \in S(c) \ \textit{such that} \ F(u) \le 0\}, \tag{2.1.3}$$

then $c_2 \in (0,\infty)$ *and*

$$\begin{cases} m(c) = 0, & \textit{as} \ \ c \in (0, c_2); \\ m(c) = -\infty, & \textit{as} \ \ c \in (c_2, \infty). \end{cases} \tag{2.1.4}$$

Our result concerning the existence or non-existence of a minimizer is

Theorem 2.1.2. *(i) When* $p \in (3, \frac{10}{3})$*,* $m(c)$ *has a minimizer if and only if* $c \in [c_1, \infty)$*.*

(ii) When $p = 3$ *or* $p = \frac{10}{3}$*,* $m(c)$ *has no minimizer for any* $c > 0$*.*

Remark 2.1.3. One always has $m(c) \leq 0$ for any $c > 0$. Indeed let $u \in S(c)$ be arbitrary and consider the scaling $u^t(x) = t^{\frac{3}{2}}u(tx)$. We have $u^t \in S(c)$ for any $t > 0$ and also

$$F(u^t) = \frac{t^2}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{t}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|} dxdy - \frac{t^{\frac{3}{2}(p-2)}}{p}\int_{\mathbb{R}^3} |u|^p dx.$$

Thus $F(u^t) \to 0$ as $t \to 0$ and the conclusion follows.

Remark 2.1.4. In [34], the authors recently study the existence of global minimizers of the functional

$$E_d(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - \frac{d}{p}\int_{\mathbb{R}^3} |u|^p dx,\ d > 0,$$

on the constraint $S(c)$. In the range $p \in (3, \frac{10}{3})$, they manage to give in term of some best Sobolev constants an “explicit” value for the threshold value $c_1 > 0$ (c_1 is given in (2.1.2)). Also when $p = 3$ they prove that there exists a $d_0 > 0$ such that for any $c > 0$, $m(c) < 0$ and the functional E_d has a global minimizer on $S(c)$ if $d > d_0$. On the contrary $m(c) = 0$ and E_d has no minimizer when $d < d_0$. This result which should be set in parallel with Theorem 2.1.1 ($\mathbb{III}$) (*iv*) and Theorem 2.1.2 (*ii*), gives a new light on the case $p = 3$. In addition, Theorem 2.1.2 (*ii*) implies that necessarily $d_0 > 1$.

Remark 2.1.5. Theorem 2.1.2 provides a fairly complete answer to the issue of global minimizers for F on $S(c)$ when $p \in [3, \frac{10}{3}]$. By contrast, when $p \in (2, 3)$, as one sees in Theorem 1.1.1, the situation is much less understood. It is only known that a minimizer exists when $c > 0$ is sufficiently small. Clearly for any $c > 0$, $m(c) < 0$ and any minimizing sequence is bounded. However in trying to develop a minimization process one faces the difficulty in ruling out the possible dichotomy of the minimizing sequences. Thus it is still an open question whether or not $m(c)$ is reached for $c > 0$ large.

In addition to the non-existence results of Theorem 2.1.2, we also show that, taking eventually $c > 0$ smaller, there are no critical points of F on $S(c)$. Precisely

Theorem 2.1.6. *When $p \in (3, \frac{10}{3}]$, there exists $\bar{c} > 0$ such that for any $c \in (0, \bar{c})$, there are no critical points of F restricted to $S(c)$. When $p = 3$, for all $c > 0$, F does not admit critical points on the constraint $S(c)$.*

Remark 2.1.7. Theorem 2.1.6 is, up to our knowledge, the only result where a non-existence result of small L^2-norm solutions is established for (E_λ). Note however that in [70, 99] it was independently proved that when $p \in (2, 3]$ there exists a $\lambda_0 < 0$ such that (E_λ) has only trivial solution when $\lambda \in (-\infty, \lambda_0)$.

2.2 Preliminary results

To obtain our non-existence results we use the fact that any critical point of F on $S(c)$ satisfies $Q(u) = 0$ where

$$Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|} dxdy - \frac{3(p-2)}{2p}\int_{\mathbb{R}^3} |u|^p dx.$$

Indeed we have

Lemma 2.2.1. *If u_0 is a critical point of F on $S(c)$, then $Q(u_0) = 0$.*

Proof. First we denote

$$T_\lambda(u) := \langle S'_\lambda(u), u\rangle = A(u) - \lambda D(u) + B(u) - C(u), \tag{2.2.1}$$
$$P_\lambda(u) := \frac{1}{2}A(u) - \frac{3}{2}\lambda D(u) + \frac{5}{4}B(u) - \frac{3}{p}C(u). \tag{2.2.2}$$

Here $\lambda \in \mathbb{R}$ is a parameter and $S_\lambda(u)$ is the energy functional corresponding to the equation (E_λ), i.e.

$$S_\lambda(u) := \frac{1}{2}A(u) - \frac{\lambda}{2}D(u) + \frac{1}{4}B(u) - \frac{1}{p}C(u). \tag{2.2.3}$$

Clearly $S_\lambda(u) = F(u) - \frac{\lambda}{2}D(u)$ and simple calculations imply that

$$\frac{3}{2}T_\lambda(u) - P_\lambda(u) = Q(u). \tag{2.2.4}$$

Now from [43] or [99, Theorem 2.2], we know that $P_\lambda(u) = 0$ is a Pohozaev identity for the Schrödinger-Poisson equation (E_λ). In particular any critical point u of $S_\lambda(u)$ satisfies $P_\lambda(u) = 0$.

On the other hand, since u_0 is a critical point of F restricted to $S(c)$, there exists a Lagrange multiplier $\lambda_0 \in \mathbb{R}$, such that

$$F'(u_0) = \lambda_0 u_0.$$

Thus for any $\phi \in H^1(\mathbb{R}^3)$,

$$\langle S'_{\lambda_0}(u_0), \phi\rangle = \langle F'(u_0) - \lambda_0 u_0, \phi\rangle = 0, \tag{2.2.5}$$

which shows that u_0 is also a critical point of $S_{\lambda_0}(u)$. Hence

$$P_{\lambda_0}(u_0) = 0, \quad T_{\lambda_0}(u_0) = \langle S'_{\lambda_0}(u_0), u_0\rangle = 0,$$

and $Q(u_0) = 0$ follows from (2.2.4). □

We now give an estimate on the nonlocal term, which is useful to control the functionals F and Q.

Lemma 2.2.2. *When $p \in [3, 4]$, there exists a constant $C > 0$, depending only on p, such that, for any $u \in S(c)$,*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy \geq -\frac{1}{16\pi}\,\|\nabla u\|_2^2 + C \cdot \frac{\|u\|_p^{\frac{p}{4-p}}}{\|\nabla u\|_2^{\frac{3(p-3)}{4-p}}\;\|u\|_2^{\frac{p-3}{4-p}}}. \tag{2.2.6}$$

Proof. Since $p \in [3, 4]$, by interpolation, we have

$$\|u\|_p^p \leq \|u\|_3^{3(4-p)}\,\|u\|_4^{4(p-3)}. \tag{2.2.7}$$

In addition, since $(|x|^{-1} * |u|^2) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solves the equation

$$-\Delta\Phi = 4\pi|u|^2 \quad \text{in } \mathbb{R}^3, \tag{2.2.8}$$

on one hand multiplying (2.2.8) by $(|x|^{-1} * |u|^2) \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ and integrating we get

$$4\pi \int_{\mathbb{R}^3} (|x|^{-1} * |u|^2)|u|^2 dx = \int_{\mathbb{R}^3} |\nabla(|x|^{-1} * |u|^2)|^2 dx. \tag{2.2.9}$$

On the other hand, multiplying (2.2.8) by $|u|$ and integrating we get for any $\eta > 0$,

$$\begin{aligned} 4\pi\eta \int_{\mathbb{R}^3} |u|^3 dx &= \eta \int_{\mathbb{R}^3} -\Delta(|x|^{-1} * |u|^2)|u| dx \\ &\leq \eta \int_{\mathbb{R}^3} \nabla(|x|^{-1} * |u|^2) \cdot \nabla |u| dx \\ &\leq \int_{\mathbb{R}^3} |\nabla(|x|^{-1} * |u|^2)|^2 dx + \frac{\eta^2}{4} \int_{\mathbb{R}^3} |\nabla u|^2 dx. \end{aligned} \tag{2.2.10}$$

Thus, taking $\eta = 1$ in (2.2.10) it follows from (2.2.9) and (2.2.10) that

$$\int_{\mathbb{R}^3} |u|^3 \, dx \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dx dy + \frac{1}{16\pi} \|\nabla u\|_2^2 . \tag{2.2.11}$$

Now, using Gagliardo-Nirenberg's inequality, there exists a constant $C > 0$, depending only on p, such that

$$\int_{\mathbb{R}^3} |u|^4 \, dx \leq C \, \|\nabla u\|_2^3 \, \|u\|_2 . \tag{2.2.12}$$

Taking (2.2.11) and (2.2.12) into (2.2.7), we obtain

$$\|u\|_p^p \leq C \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dx dy + \frac{1}{16\pi} \|\nabla u\|_2^2 \right)^{(4-p)} \|\nabla u\|_2^{3(p-3)} \, \|u\|_2^{(p-3)} ,$$

which implies (2.2.6). $\square$

The estimate (2.2.6) leads to a lower bound on $Q(u)$.

Lemma 2.2.3. *When $p \in (3, \frac{10}{3})$, there exists a constant $C > 0$, depending only on p, such that, for any $u \in S(c)$*

$$Q(u) \geq \frac{64\pi - 1}{64\pi} A(u) - C \cdot A(u)^{\frac{3}{2}} \cdot c^{\frac{1}{2}}. \tag{2.2.13}$$

Proof. By Lemma 2.2.2 there exists a constant $C > 0$ depending only on p, such that, for any $u \in S(c)$,

$$Q(u) \geq \frac{64\pi - 1}{64\pi} A(u) + C \cdot \frac{C(u)^{\frac{1}{4-p}}}{A(u)^{\frac{3(p-3)}{2(4-p)}} \cdot D(u)^{\frac{p-3}{2(4-p)}}} - \frac{3(p-2)}{2p} C(u). \tag{2.2.14}$$

To obtain (2.2.13) from (2.2.14) we introduce the auxiliary function

$$f_K(x) = \left(\frac{64\pi - 1}{64\pi} \right) K + D \cdot x^{\frac{1}{4-p}} - \frac{3(p-2)}{2p} \cdot x, \quad x > 0$$

with $D = C \cdot \left(K^{\frac{3(p-3)}{2(4-p)}} \cdot c^{\frac{p-3}{2(4-p)}} \right)^{-1}$. Its study will provide us an estimate independent of $C(u)$. Clearly

$$f'_K(x) = D \cdot \frac{1}{4-p} \cdot x^{\frac{p-3}{4-p}} - \frac{3(p-2)}{2p},$$

$$f_K''(x) \quad = \quad D \cdot \frac{1}{4-p} \cdot \frac{p-3}{4-p} \cdot x^{\frac{p-3}{4-p}-1} > 0, \qquad \text{for all } x > 0.$$

Therefore $f_K(x)$ has the unique global minimum at

$$\bar{x} = \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{4-p}{p-3}},$$

and

$$\begin{aligned} f_K(\bar{x}) &= \frac{64\pi - 1}{64\pi} K + D \cdot \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{1}{p-3}} - \frac{3(p-2)}{2p} \cdot \left(\frac{3(p-2)(4-p)}{2pD} \right)^{\frac{4-p}{p-3}} \\ &= \frac{64\pi - 1}{64\pi} K - \left(\frac{3(p-2)(4-p)}{2p} \right)^{\frac{1}{p-3}} \cdot \frac{p-3}{4-p} \cdot D^{\frac{p-4}{p-3}} \\ &= \frac{64\pi - 1}{64\pi} K - \left(\frac{3(p-2)(4-p)}{2p} \right)^{\frac{1}{p-3}} \cdot \frac{p-3}{4-p} \cdot C^{\frac{p-4}{p-3}} \cdot K^{\frac{3}{2}} \cdot c^{\frac{1}{2}}. \end{aligned}$$

Thus $f_K(x) \geq f_K(\bar{x})$ for all $x > 0$. This, together with (2.2.14) implies (2.2.13). □

Finally we recall the following results obtained in [14, 15].

Lemma 2.2.4. *Let $p \in (3, \frac{10}{3})$, then*

(i) For any $c > 0$ such that $m(c) < 0$, $m(c)$ admits a minimizer.

(ii) There exists $d > 0$, such that for all $c \in (d, \infty)$, $m(c) < 0$.

(iii) The function $c \to m(c)$ is continuous at each $c > 0$.

Remark 2.2.5. Points (i) and (ii) of Lemma 2.2.4 are proved in [14]. Concerning Point (iii), in [15] the authors prove the continuity of $m(c)$ about $c > 0$ when $p \in (2, 3)$. However inspecting their proof reveals that it also holds for $p \in [3, \frac{10}{3})$.

2.3 Proofs of the main results

We first give the following non-existence result.

Lemma 2.3.1. *When $p \in (3, \frac{10}{3})$, there exists a $c_3 > 0$, such that $m(c)$ has no minimizer for all $c \in (0, c_3)$.*

Proof. Let us assume by contradiction that there exist sequences $\{c_n\} \subset \mathbb{R}^+$, with $c_n \to 0$ as $n \to \infty$, and $\{u_n\} \subset S(c_n)$ such that $F(u_n) = m(c_n)$. Then by Lemma 2.2.1, $Q(u_n) = 0$ for any $n \in \mathbb{N}$.

Since $m(c) \leq 0$ for any $c > 0$, see Remark 2.1.3, we know that $F(u_n) \leq 0$. Thus

$$\begin{aligned} \frac{1}{2} A(u_n) + \frac{1}{4} B(u_n) &\leq \frac{1}{p} C(u_n) \\ &\leq \frac{C}{p} A(u_n)^{\frac{3}{4}(p-2)} \cdot D(u_n)^{\frac{6-p}{4}}, \end{aligned} \tag{2.3.1}$$

by Gagliardo-Nirenberg's inequality. Since $p \in (3, \frac{10}{3})$, $1 > \frac{3}{4}(p-2)$ and thus (2.3.1) implies that

$$A(u_n) \to 0, \text{ as } n \to \infty. \tag{2.3.2}$$

Now due to (2.3.2) and Lemma 2.2.3, when $n \in \mathbb{N}$ is sufficiently large,

$$\begin{aligned} Q(u_n) &\geq \frac{64\pi - 1}{64\pi} A(u_n) - C \cdot A(u_n)^{\frac{3}{2}} \cdot c_n^{\frac{1}{2}} \\ &\geq \frac{64\pi - 1}{64\pi} A(u_n) - C \cdot A(u_n)^{\frac{3}{2}} > 0. \end{aligned}$$

Obviously this contradicts Lemma 2.2.1 and this ends the proof. □

The following lemma is crucial to establish a precise threshold between existence and non-existence.

Lemma 2.3.2. *Assume that $p \in (3, \frac{10}{3})$ holds. For any $c > 0$ such that $m(c) < 0$ or such that $m(c) = 0$ and $m(c)$ has a minimizer we have*

$$m(tc) < tm(c), \ \textit{for all}\ t > 1.$$

Proof. By Lemma 2.2.4 (i) without restriction we can assume that $m(c) \leq 0$ admit a minimizer $u_c \in S(c)$. We set $(u_c)_t(x) = t^2 u_c(tx)$ for $t > 1$. Then $D((u_c)_t) = tD(u_c) = tc$, and since $2p - 6 > 0$ in case of $p \in (3, 10/3]$ and $C(u_c) > 0$, we obtain

$$\begin{aligned} m(tc) \leq F((u_c)_t) &= t^3 \cdot \left(\frac{1}{2} A(u_c) + \frac{1}{4} B(u_c) - \frac{t^{2p-6}}{p} C(u_c) \right) \\ &< t^3 \cdot \left(\frac{1}{2} A(u_c) + \frac{1}{4} B(u_c) - \frac{1}{p} C(u_c) \right) \\ &= t^3 \cdot F(u_c) = t^3 m(c). \end{aligned} \tag{2.3.3}$$

Since $m(c) \leq 0$ and $t > 1$, we conclude from (2.3.3) that $m(tc) < t^3 m(c) \leq tm(c)$. □

In the case $p = \frac{10}{3}$, we first have

Lemma 2.3.3. *When $p = \frac{10}{3}$, we have $c_2 \in (0, \infty)$, where c_2 is given by (2.1.3).*

Proof. First observe that by Gagliardo-Nirenberg's inequality, when $p = \frac{10}{3}$ we have

$$C(u) \leq C \cdot A(u) \cdot c^{\frac{2}{3}}, \qquad \text{for all } u \in S(c), \tag{2.3.4}$$

where $C > 0$ independent of $c > 0$. Thus for any $u \in S(c)$, there holds

$$\begin{aligned} F(u) &\geq \frac{1}{2} A(u) + \frac{1}{4} B(u) - \frac{3}{10} C \cdot A(u) \cdot c^{\frac{2}{3}} \\ &\geq A(u) \left(\frac{1}{2} - \frac{3}{10} C \cdot c^{\frac{2}{3}} \right). \end{aligned} \tag{2.3.5}$$

Thus $F(u) > 0$, for all $u \in S(c)$ if $c > 0$ is sufficiently small and it proves that $c_2 > 0$.

Now take $u_1 \in S(1)$ arbitrary and consider the scaling

$$u_t(x) = t^2 u_1(tx), \qquad \text{for all } t > 0. \tag{2.3.6}$$

Then $u_t \in S(t)$ and

$$\begin{aligned} F(u_t) &= \frac{t^3}{2} A(u_1) + \frac{t^3}{4} B(u_1) - \frac{3}{10} t^{\frac{11}{3}} C(u_1) \\ &= t^3 \left(\frac{1}{2} A(u_1) + \frac{1}{4} B(u_1) - \frac{3}{10} t^{\frac{2}{3}} C(u_1) \right). \end{aligned} \tag{2.3.7}$$

This shows that $F(u_t) < 0$ for $t > 0$ large enough and proves that $c_2 < \infty$. □

We can now give the

Proof of Theorem 2.1.1. First we prove that $c_1 > 0$ by contradiction. If we assume that $c_1 = 0$ then, from the definition of c_1, $m(c) < 0$ for all $c > 0$. Thus Lemma 2.2.4 (i) implies the existence of a minimizer for any $c > 0$ and this contradicts Lemma 2.3.1. Additionally Lemma 2.2.4 (ii) shows that $c_1 < \infty$, thus Point (i) follows. To prove Point (ii) we observe that since $m(c) \leq 0$ for all $c > 0$, from the definition of $c_1 > 0$ it follows that $m(c) = 0$ if $c \in (0, c_1)$. Using the continuity of $c \mapsto m(c)$, see Lemma 2.2.4 (iii), we obtain that $m(c_1) = 0$ and then Point (ii) holds. Point (iii) is a direct consequence of Lemma 2.3.2 and of the definition of $c_1 > 0$.

Concerning Point (iv), it is enough to show that if $p = 3$, for any $c > 0$ one has

$$F(u) > 0, \quad \text{for all } u \in S(c). \tag{2.3.8}$$

Indeed, since $m(c) \leq 0$ for all $c > 0$, (2.3.8) implies immediately Point (iv). To check (2.3.8), we use (2.2.10) with $\eta = 4/3$. From (2.2.9) and (2.2.10) we then get

$$\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \geq -\frac{1}{36\pi}||\nabla u||_2^2 + \frac{1}{3}||u||_3^3.$$

Thus when $p = 3$, for any $u \in S(c)$,

$$F(u) \geq \frac{1}{2}||\nabla u||_2^2 - \frac{1}{36\pi}||\nabla u||_2^2 > 0$$

and (2.3.8) holds.

Finally since, by Lemma 2.3.3, $c_2 \in (0, \infty)$, to prove Point (v) it is enough to verify (2.1.4). From the definition of c_2, it follows directly that $m(c) = 0$ for any $c \in (0, c_2)$. Now if $c \in (c_2, \infty)$, we first claim that there exists a $v \in S(c)$ such that $F(v) \leq 0$. Indeed if we assume that $F(u) > 0$ for all $u \in S(c)$ we reach a contradiction as follows. For an arbitrary $\hat{c} \in [c_2, c)$ taking any $u \in S(\hat{c})$ we scale it as in (2.3.6) where $t = c/\hat{c}$. Then $u_t \in S(c)$ and it follows from (2.3.7) that $F(u_t) \leq t^3 F(u)$. This implies that $F(u) > 0$ for all $u \in S(\hat{c})$ and since $\hat{c} \in [c_2, c)$ is arbitrary this contradicts the definition of $c_2 > 0$. Hence, for any $c \in (c_2, \infty)$, there exists a $u_0 \in S(c)$ such that $F(u_0) \leq 0$.

Consider now the scaling

$$u^\theta(x) := \theta^{\frac{3}{2}} u_0(\theta x), \quad \text{for all } \theta > 0. \tag{2.3.9}$$

We have $u^\theta \in S(c)$ for all $\theta > 0$ and

$$\begin{aligned} F(u^\theta) &= \frac{\theta^2}{2}A(u_0) + \frac{\theta}{4}B(u_0) - \frac{10}{3}\theta^2 C(u_0) \\ &= \frac{\theta}{4}B(u_0) - \Big(\frac{10}{3}C(u_0) - \frac{1}{2}A(u_0)\Big)\cdot\theta^2. \end{aligned} \tag{2.3.10}$$

Since $F(u_0) \leq 0$, necessarily

$$\frac{10}{3}C(u_0) - \frac{1}{2}A(u_0) > 0.$$

Thus we see from (2.3.10) that $\lim_{\theta\to\infty} F(u^\theta) = -\infty$ and $m(c) = -\infty$ follows. At this point the proof of the theorem is completed. □

Before giving the proof of Theorem 2.1.2 we consider the case where $c = c_1$ that requires a special treatment.

Lemma 2.3.4. *Assume that $p \in (3, \frac{10}{3})$ holds. Then $m(c_1)$ admits a minimizer.*

Proof. Let $k_n := c_1 + 1/n$, for all $n \in \mathbb{N}$. We have $k_n \to c_1$ and thus, by Lemma 2.2.4 (*iii*), $m(k_n) \to m(c_1) = 0$. Furthermore, by Theorem 2.1.1 (*iii*) and Lemma 2.2.4 (*i*) we know that for each $n \in \mathbb{N}$, $m(k_n) < 0$ and $m(k_n)$ admits a minimizer u_n. Now we claim that the sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$. Indeed, by Gagliardo-Nirenberg's inequality, we have

$$\begin{aligned} \frac{1}{2}A(u_n) + \frac{1}{4}B(u_n) &= \frac{1}{p}C(u_n) + F(u_n) \\ &\leq CA(u_n)^{\frac{3(p-2)}{4}} k_n^{\frac{6-p}{4}} + m(k_n). \end{aligned}$$

This implies that $\{A(u_n)\}$ is bounded, since $m(k_n) \leq 0$ and $1 > 3(p-2)/4$. Thus we conclude that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$.

Now we claim that $C(u_n) \nrightarrow 0$. By contradiction let us assume that $C(u_n) \to 0$ as $n \to \infty$. Since $F(u_n) \to m(c_1) = 0$ it then follows that

$$A(u_n) \to 0 \ \text{ and } \ B(u_n) \to 0, \text{ as } \ n \to \infty. \tag{2.3.11}$$

Now, similarly to the proof of Lemma 2.2.3, using (2.2.6), we can estimate $F(u)$ from below by

$$F(u) \geq \frac{32\pi - 1}{64\pi} A(u) - C \cdot A(u)^{\frac{3}{2}} \cdot c^{\frac{1}{2}}, \qquad \text{for all } u \in S(c) \tag{2.3.12}$$

where $C > 0$ is constant, depending only on p. In particular

$$F(u_n) \geq A(u_n) \left(\frac{32\pi - 1}{64\pi} - C \cdot A(u_n)^{\frac{1}{2}} \cdot k_n^{\frac{1}{2}} \right). \tag{2.3.13}$$

Taking (2.3.11) into account, (2.3.13) implies that $F(u_n) \geq 0$ for $n \in \mathbb{N}$ sufficiently large. This contradicts the fact that $F(u_n) = m(k_n) < 0$ for all $n \in \mathbb{N}$ and proves the claim.

Now, by [83, Lemma I.1], we deduce that $\{u_n\}$ does not vanish. Namely that there exists a constant $\delta > 0$ and a sequence $\{x_n\} \subset \mathbb{R}^3$ such that

$$\int_{B(x_n,1)} |u_n|^2 dx \geq \delta > 0,$$

or equivalently

$$\int_{B(0,1)} |u_n(\cdot + x_n)|^2 dx \geq \delta > 0. \tag{2.3.14}$$

Now let $v_n(\cdot) := u_n(\cdot + x_n)$. Clearly $\{v_n\}$ is bounded in $H^1(\mathbb{R}^3)$ and thus there exists $v_0 \in H^1(\mathbb{R}^3)$ such that

$$v_n \rightharpoonup v_0 \ \text{ weakly in } H^1(\mathbb{R}^3) \quad \text{ and } \quad v_n \to v_0 \ \text{ in } L^2_{loc}(\mathbb{R}^3).$$

We note that $v_0 \neq 0$, since by (2.3.14)

$$0 < \delta \leq \lim_{n\to\infty} \int_{B(0,1)} |v_n|^2 dx = \int_{B(0,1)} |v_0|^2 dx.$$

Let us prove that v_0 is a minimizer of $m(c_1)$. First we show that $F(v_0) = 0$. Clearly

$$\lim_{n\to\infty} \|v_n\|_2^2 = \|v_0\|_2^2 + \lim_{n\to\infty} \|v_n - v_0\|_2^2 = c_1 \tag{2.3.15}$$

and using Lemma 2.2.4 (*iii*) we deduce from (2.3.15) that

$$\lim_{n\to\infty} F(v_n - v_0) \geq \lim_{n\to\infty} m(||v_n - v_0||_2^2) = m(c_1 - ||v_0||_2^2) = 0. \tag{2.3.16}$$

Here we make the convention that $m(0) = 0$. Now using [112, Lemma 2.2], we have

$$0 = m(c_1) = \lim_{n\to\infty} F(v_n) = F(v_0) + \lim_{n\to\infty} F(v_n - v_0). \tag{2.3.17}$$

Since $||v_0||_2^2 \leq c_1$ we have $m(||v_0||_2^2) = 0$ and it shows that $F(v_0) < 0$ is impossible. From (2.3.16) and (2.3.17) we deduce that $F(v_0) = 0$ and that v_0 is a minimizer associated with $m(||v_0||_2^2)$. If we assume that $||v_0||_2^2 < c_1$ we get a contradiction with Lemma 2.3.2 since $m(c_1) = 0$. Thus necessarily $\|v_0\|_2^2 = c_1$ and this ends the proof. □

Proof of Theorem 2.1.2. To prove Point (*i*) we assume by contradiction that there exists $\widetilde{c} \in (0, c_1)$ such that $m(\widetilde{c})$ admits a minimizer. Then from the definition of $c_1 > 0$ we get that $m(\widetilde{c}) = 0$ and Lemma 2.3.2 implies that $m(c) < 0$ for any $c > \widetilde{c}$. This contradicts the definition of $c_1 > 0$. Now when $c > c_1$ the result clearly follows from Theorem 2.1.1 (*iii*) and Lemma 2.2.4 (*i*). Finally the case $c = c_1$ is considered in Lemma 2.3.4. For Point (*ii*), first observe that, because of (2.3.8), when $p = 3$, for any $c > 0$, $m(c)$ does not have a minimizer. Then we note that, from the definition of $Q(u)$, it holds, for any $u \in S(c)$,

$$F(u) - \frac{2}{3(p-2)} Q(u) = \frac{3p-10}{6(p-2)} A(u) + \frac{3p-8}{12(p-2)} B(u). \tag{2.3.18}$$

Taking $p = \frac{10}{3}$ in (2.3.18) we obtain

$$F(u) - \frac{1}{2} Q(u) = \frac{1}{8} B(u). \tag{2.3.19}$$

Thus if we assume by contradiction that $m(c)$ has a minimizer $u_c \in S(c)$ for some $c > 0$ we see from Lemma 2.2.1 and (2.3.19) that

$$0 \geq m(c) = F(u_c) = \frac{1}{8} B(u_c) > 0.$$

This contradiction ends the proof of Point (*ii*) and of the theorem. □

Proof of Theorem 2.1.6. We first consider the case $p \in (3, \frac{10}{3}]$ and we assume by contradiction that there exists sequences $\{c_n\} \subset \mathbb{R}^+$, with $c_n \to 0$, as $n \to \infty$, and $\{u_n\} \subset S(c_n)$ such that $u_n \in S(c_n)$ is a critical point of F restricted to $S(c_n)$. Then since

$$Q(u_n) = A(u_n) + \frac{1}{4} B(u_n) - \frac{3(p-2)}{2p} C(u_n) = 0,$$

we deduce, from Gagliardo-Nirenberg's inequality, that for some $C > 0$,

$$A(u_n) \leq \frac{3(p-2)}{2p} C(u_n) \leq C \cdot A(u_n)^{\frac{3(p-2)}{4}} \cdot c_n^{\frac{6-p}{4}}. \tag{2.3.20}$$

Thus there holds

$$A(u_n)^{\frac{10-3p}{4}} \leq C \cdot c_n^{\frac{6-p}{4}}$$

and we get that

$$A(u_n) \to 0 \quad \text{as} \quad n \to \infty \tag{2.3.21}$$

if $p \in (3, \frac{10}{3})$ and directly a contradiction if $p = \frac{10}{3}$. Now when $p \in (3, \frac{10}{3})$ by Lemma 2.2.3 we know, since $Q(u_n) = 0$, that there exists a constant $C > 0$ such that

$$\frac{64\pi - 1}{64\pi} A(u_n) \leq C \cdot A(u_n)^{\frac{3}{2}} \cdot c_n^{\frac{1}{2}}$$

or equivalently that

$$\frac{64\pi - 1}{64\pi} \leq C \cdot A(u_n)^{\frac{1}{2}} \cdot c_n^{\frac{1}{2}}. \tag{2.3.22}$$

But (2.3.22) implies that $A(u_n) \to \infty$ as $n \to \infty$ and this contradicts (2.3.21).

Now when $p = 3$, it is enough to prove that, for any $c > 0$, there holds

$$Q(u) > 0, \qquad \text{for all } u \in S(c). \tag{2.3.23}$$

Indeed, if (2.3.23) holds true, we can conclude the non-existence of minimizers directly from Lemma 2.2.1. To check (2.3.23), we use (2.2.10) with $\eta = 2$. Then, from (2.2.9) and (2.2.10), we get

$$\frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy \geq -\frac{1}{16\pi}||\nabla u||_2^2 + \frac{1}{2}||u||_3^3.$$

Thus, for any $u \in S(c)$,

$$\begin{aligned} Q(u) &= ||\nabla u||_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - \frac{1}{2}||u||_3^3 \\ &\geq ||\nabla u||_2^2 - \frac{1}{16\pi}||\nabla u||_2^2 > 0. \end{aligned}$$

At this point the proof is completed. □

Chapter 3

Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson-Slater equations

3.1 Introduction

In this chapter we prove the existence and the strong instability of standing waves with a prescribed L^2-norm for the following Schrödinger-Poisson-Slater equations:

$$i\partial_t\varphi + \Delta\varphi - (|x|^{-1} * |\varphi|^2)\varphi + |\varphi|^{p-2}\varphi = 0 \ \text{ in } \ \mathbb{R}\times\mathbb{R}^3. \tag{3.1.1}$$

This class of Schrödinger type equations with a repulsive nonlocal Coulombic potential is obtained by approximation of the Hartree-Fock equation describing a quantum mechanical system of many particles, see for instance [10, 81, 84, 92]. We look for standing waves solutions of (3.1.1). Namely for solutions in the form

$$\varphi(t,x) = e^{-i\lambda t}u(x),$$

where $\lambda \in \mathbb{R}$ is a parameter. Then the function $u(x)$ satisfies the equation

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0 \ \text{ in } \ \mathbb{R}^3. \tag{E_λ}$$

Thus to that purpose, one of the aims is, for given $c > 0$, to search for

$$(u_c, \lambda_c) \in H^1(\mathbb{R}^3)\times\mathbb{R} \ \text{ solution of } \ (E_\lambda) \text{ with } \|u_c\|_2^2 = c.$$

One notes that a solution u_c of (E_λ) can be obtained as constrained critical points of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2\,|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p\,dx$$

on the constraint

$$S(c) := \Big\{u \in H^1(\mathbb{R}^3):\ \|u\|_2^2 = c,\ c > 0\Big\}.$$

The parameter $\lambda_c \in \mathbb{R}$ in this situation appears as a Lagrange multiplier in (E_λ).

We recall that the functional F is a well defined and C^1 functional on $S(c)$ for any $p \in (2, 6]$ (see [99] for example), and when $p \in (2, \frac{10}{3})$, F is bounded from below and coercive on $S(c)$, which permits to obtain critical points of F on $S(c)$ by considering global minimizers of F on the constraint. We refer to Chapter 2 for more details in that direction.

In this chapter, we consider the case $p \in (\frac{10}{3}, 6)$. For this range of power the functional F is no more bounded from below on $S(c)$. No hope can be expected to find a solution as a global minimizer for F on $S(c)$. However, we shall prove that the constrained functional has a mountain pass geometry.

Definition 3.1.1. Given $c > 0$, we say that F has a mountain pass geometry on $S(c)$ if there exists a $K_c > 0$, such that

$$\gamma(c) := \inf_{g \in \Gamma_c} \max_{t \in [0,1]} F(g(t)) > \max\Big\{ \max_{g \in \Gamma_c} F(g(0)), \max_{g \in \Gamma_c} F(g(1)) \Big\},$$

holds in the set

$$\Gamma_c = \Big\{ g \in C([0,1], S(c)) : \ g(0) \in A_{K_c}, F(g(1)) < 0 \Big\},$$

where $A_{K_c} = \{u \in S(c) : \ \|\nabla u\|_2^2 \le K_c\}$.

In order to find critical points of F on $S(c)$, we look at the mountain pass level $\gamma(c)$. Our main result concerning the existence of solutions of (E_λ) is given by the following

Theorem 3.1.2. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$, then F has a mountain pass geometry on $S(c)$. Moreover there exists $c_0 > 0$ such that for any $c \in (0, c_0)$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}^-$ solution of (E_λ) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$.*

In addition, we prove that

Lemma 3.1.3. *Assume that $p \in (\frac{10}{3}, 6)$. Let (u_c, λ_c) with $\|u_c\|_2^2 = c$, be a solution of (E_λ). Then necessarily*

$$\lambda_c \to -\infty, \ \text{as } c \to 0. \tag{3.1.2}$$

Now let us underline some of the difficulties that arise in the study of the existence of critical points for our functional on $S(c)$. First the mountain pass geometry does not guarantee the existence of a bounded Palais-Smale sequence. To overcome this difficulty we introduce the functional

$$Q(u) := \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{1}{4} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 \, |u(y)|^2}{|x-y|} dxdy - \frac{3(p-2)}{2p} \int_{\mathbb{R}^3} |u|^p dx,$$

the set

$$V(c) := \Big\{ u \in S(c) \ : \ Q(u) = 0 \Big\}$$

and we first prove that

$$\gamma(c) = \inf_{u \in V(c)} F(u) > 0. \tag{3.1.3}$$

We also show that each constrained critical point of F must lie in $V(c)$. At this point, taking advantage of the nice "shape" of some sequence of paths $\{g_n\} \subset \Gamma_c$ such that

$$\max_{t\in[0,1]} F(g_n(t)) \to \gamma(c),$$

we construct a special Palais-Smale sequence $\{u_n\} \subset S(c)$ at the level $\gamma(c)$, which concentrates around $V(c)$. This localization leads to its boundedness but also provides the information that $Q(u_n) = o(1)$. This last property is crucially used in the study of the compactness of the sequence. Next, since we look for solutions with a prescribed L^2-norm, we must deal with a possible lack of compactness for sequences which do not minimize globally F on $S(c)$. In our setting, it seems not be possible to reduce the problem to the classical vanishing-dichotomy-compactness scenario and to the check of the associated strict subadditivity inequalities, see [83]. To overcome this difficulty, we first study the behavior of the function $c \to \gamma(c)$. The theorem below summarizes its properties.

Theorem 3.1.4. *Let $p \in (\frac{10}{3}, 6)$ and for any $c > 0$ let $\gamma(c)$ be the mountain pass level. Then*

(i) $c \to \gamma(c)$ *is continuous at each* $c > 0$.

(ii) $c \to \gamma(c)$ *is non-increasing.*

(iii) There exists $c_0 > 0$ *such that in* $(0, c_0)$ *the function* $c \to \gamma(c)$ *is strictly decreasing.*

(iv) There exists $c_\infty > 0$ *such that for all* $c \geq c_\infty$ *the function* $c \to \gamma(c)$ *is constant.*

(v) $\lim_{c\to 0} \gamma(c) = +\infty$ *and* $\lim_{c\to\infty} \gamma(c) =: \gamma(\infty) > 0$.

We show that if $\gamma(c) < \gamma(c_1)$, for all $c_1 \in (0, c)$ then there exists a $u_c \in H^1(\mathbb{R}^3)$ such that $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. However we are only able to prove this for $c > 0$ sufficiently small. For the other values of $c > 0$, the information that $c \to \gamma(c)$ is non increasing permits to reduce the problem of convergence to the one of showing that the associated Lagrange multiplier $\lambda_c \in \mathbb{R}$ is non zero. But we do prove that $\lambda_c = 0$ holds for any $c > 0$ is sufficiently large. In view of this point, we conjecture that $\gamma(c)$ is not a critical value for $c > 0$ large enough. See Remark 3.7.4 in that direction.

Remark 3.1.5. The proof that $c \to \gamma(c)$ is non increasing is not derived through the use of some scaling. Due to the presence of three terms in $F(u)$ which scale differently such an approach seems difficult. Instead we show that if one adds in a suitable way L^2-norm in $\mathbb{R}^3$ then this does not increase the mountain pass level. This approach is reminiscent of the one developed in [67] but here the fact that we deal with a function defined by a mountain pass instead of a global minimum and that $F(u)$ has a nonlocal term makes the proof more delicate.

To show Theorem 3.1.4 *(iv)* and that $\gamma(c) \to \gamma(\infty) > 0$ as $c \to \infty$ in *(v)* we take advantage of some results of [63]. In [63] the equation

$$-\Delta v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \ \text{ in } \ \mathbb{R}^3 \tag{3.1.4}$$

is considered. Real solutions of (3.1.4) are searched in the space

$$E := \left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy < \infty \right\} \tag{3.1.5}$$

which contains $H^1(\mathbb{R}^3)$. This space is the natural space when $\lambda = 0$ in (E_λ). In [63] it is shown that F defined in E possess a ground state. It is also proved, see [63, Theorem 6.1], that any real radial solution of (3.1.4) decreases exponentially at infinity. We extend here this result to any real solution of (3.1.4). More precisely we prove

Theorem 3.1.6. *Let $p \in (3,6)$ and $(u,\lambda) \in E \times \mathbb{R}$ with $\lambda \leq 0$ be a real solution of (E_λ). Then there exist constants $C_1 > 0$, $C_2 > 0$ and $R > 0$ such that*

$$|u(x)| \leq C_1 |x|^{-\frac{3}{4}} e^{-C_2\sqrt{|x|}}, \ \forall \ |x| > R. \tag{3.1.6}$$

In particular, $u \in H^1(\mathbb{R}^3)$.

Remark 3.1.7. Clearly the difficult case here is when $\lambda = 0$ and it corresponds to the so-called *zero mass case,* see [19]. This part of Theorem 3.1.6 was kindly provided to us by L. Dupaigne [47]. We point out that the exponential decay when $\lambda = 0$ is due to the fact that the nonlocal term is sufficiently strong at infinity. Actually we prove that $(|x|^{-1} * |v|^2) \geq C|x|^{-1}$ for some $C > 0$ and $|x|$ large. In contrast we recall that for the equation

$$-\Delta u + V(x)u - |u|^{p-2}u = 0, \quad x \in H^1(\mathbb{R}^3), \tag{3.1.7}$$

if we assume that $\limsup_{|x|\to\infty} V(x)|x|^{2+\delta} = 0$ for some $\delta > 0$, then positive solutions of (3.1.7) decay no faster than $|x|^{-1}$. This can be seen by comparing with an explicit subsolution at infinity $|x|^{-1}(1 + |x|^{-\delta})$ of $-\Delta + V$.

Theorem 3.1.6 is interesting for itself and also it answers a conjecture of [63, Remark 6.2]. For our study the information that any solution of (3.1.4) belongs to $L^2(\mathbb{R}^3)$ is crucial to derive Theorem 3.1.4 (iv)-(v) and the exponential decay is also used later to prove that our solutions correspond to standing waves unstable by blow-up.

The fact that $c \to \gamma(c)$ becomes constant for $c > 0$ large (which leads very likely to the fact that $\gamma(c)$ is not a critical level for $c > 0$ large) is due to the term $(|x|^{-1} * |u|^2)u$. In order to try to understand this, we draw a comparison between (3.1.1) and the classical nonlinear Schrödinger equation

$$i\psi_t + \Delta\psi + |\psi|^{p-2}\psi = 0 \quad \text{in } \mathbb{R}^3. \tag{3.1.8}$$

In [64], L. Jeanjean considered the existence of standing waves for (3.1.8) on $S(c)$ when $p \in [\frac{10}{3}, 6)$. Then the associated functional is unbounded from below. In [64] a solution was obtained for any given $c > 0$ after having shown that the associated Lagrange multiplier is strictly negative for any $c > 0$. In this thesis we complement and enlighten this result by showing that the mountain pass value $\tilde{\gamma}(c)$ associated with (3.1.8) is strictly decreasing as a function of $c > 0$ and that $\tilde{\gamma}(c) \to 0$ as $c \to \infty$.

The fact that (3.1.3) holds and that any constrained critical point of F lies in $V(c)$ implies that the solutions found in Theorem 3.1.2 can be considered as ground-states within the solutions having the same L^2-norm.

Let us denote the set of minimizers of F on $V(c)$ as

$$\mathcal{M}_c := \{u_c \in V(c) \ : \ F(u_c) = \inf_{u \in V(c)} F(u)\}. \tag{3.1.9}$$

Then we prove

Theorem 3.1.8. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$ there exists a $\lambda_c \leq 0$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves (E_λ).*

Clearly to prove Theorem 3.1.8 we need to show that any minimizer of F on $V(c)$ is a critical point of F restricted to $S(c)$, namely that $V(c)$ acts as a natural constraint. As additional properties of elements of $\mathcal{M}_c$, we have :

Lemma 3.1.9. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$ be arbitrary. Then*

(i) If $u_c \in \mathcal{M}_c$ then also $|u_c| \in \mathcal{M}_c$.

(ii) Any minimizer $u_c \in \mathcal{M}_c$ has the form $e^{i\theta}|u_c|$ for some $\theta \in \mathbb{S}^1$ and $|u_c(x)| > 0$ a.e. in $\mathbb{R}^3$.

In view of Lemma 3.1.9 each elements of $\mathcal{M}_c$ is a real positive function multiply by a constant complex factor.

Remark 3.1.10. A natural question that arises, as a consequence of Theorem 3.1.8, is why not search for solutions of (E_λ) with a prescribed norm by directly minimizing F on $V(c)$. However starting from an arbitrary minimizing sequence $\{u_n\} \subset V(c)$ and trying to show its convergence seem to be challenging. From the definition of $V(c)$ it is easy to prove that any minimizing sequence is bounded in $H^1(\mathbb{R}^3)$ and thus we can assume that $u_n \rightharpoonup \bar{u}$ in $H^1(\mathbb{R}^3)$ for some $\bar{u} \in H^1(\mathbb{R}^3)$. Also ruling out the vanishing is not a problem as it can be seen from Lemma 3.4.2. But to show that the dichotomy does not occur it seems necessary to know that $\bar{u} \in V(||\bar{u}||_2^2)$. For our Palais-Smale sequence we use, in Lemma 3.4.4, the information that $\bar{u} \in H^1(\mathbb{R}^3)$ is a non-trivial solution of (E_λ). Then by Lemma 3.4.3, $Q(\bar{u}) = 0$ and $\bar{u} \in V(||\bar{u}||_2^2)$. For an arbitrary minimizing sequence it does not seem possible to show that the weak limit $\bar{u} \in H^1(\mathbb{R}^3)$ belongs to $V(||\bar{u}||_2^2)$. Having such information seems to require some information on the derivative of F along the sequence and that is why we introduce Palais-Smale sequences to solve our minimization problem.

Concerning the dynamics we first consider the question of global existence of solutions for the Cauchy problem. In the case $p \in (2, \frac{10}{3})$ global existence in time is guaranteed for initial data in $H^1(\mathbb{R}^3)$, see for instance [37]. In the case $p \in (2, \frac{10}{3})$ the standing waves found in [14, 15, 102] by minimization are orbitally stable. This is proved following the approach of Cazenave-Lions [36]. In the case $p \in (\frac{10}{3}, 6)$ the global existence in time of solutions for the Cauchy problem associated with (3.1.1) does not hold for arbitrary initial condition. However we are able to prove the following global existence result.

Theorem 3.1.11. *Let $p \in (\frac{10}{3}, 6)$ and $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ be an initial condition associated with* (3.1.1) *with $c = ||u_0||_2^2$. If*

$$Q(u_0) > 0 \textit{ and } F(u_0) < \gamma(c),$$

then the solution of (3.1.1) *with initial condition u_0 exists globally in time.*

In Remark 3.8.2 we prove that the set

$$\mathcal{O} = \{u_0 \in S(c) \ : \ Q(u_0) > 0 \text{ and } F(u_0) < \gamma(c)\}$$

is not empty.

Theorem 3.1.11 is, at its modest level, in the spirit of recent works [48, 60, 93, 94, 109] which try to understand deeply the dynamics of some nonlinear equations.

In what follows, we prove that the standing waves corresponding to elements of $\mathcal{M}_c$ are strongly unstable.

Theorem 3.1.12. *Let $p \in (\frac{10}{3}, 6)$ and $c > 0$. For each $u_c \in \mathcal{M}_c$, the standing wave $e^{-i\lambda_c t}u_c$ of* (3.1.1) *where $\lambda_c \in \mathbb{R}$ is the Lagrange multiplier, is strongly unstable.*

Remark 3.1.13. In view of (3.1.3), Theorem 3.1.12 yields the strong instability of the standing waves we obtained in Theorem 3.1.2. The proof of Theorem 3.1.12 borrows elements of the original approach of H. Berestycki and T. Cazenave [18]. The starting point is the variational characterization of $u_c \in \mathcal{M}_c$ and the decay estimates established in Theorem 3.1.6 proves crucial to use the virial identity.

Remark 3.1.14. For previous results concerning the instability of standing waves of (3.1.1) we refer to [72] (see also [71]). In [72], working in the subspace of radially symmetric functions, it is proved that for $\lambda < 0$ fixed and $p \in (\frac{10}{3}, 6)$ the equation (E_λ) admits a ground state which is strongly unstable. However when we work in all $H^1(\mathbb{R}^3)$ it is still not known if ground states, or at least one of them, are radially symmetric. In that direction we are only aware of the result of [52] which gives a positive answer when $p \in (2, 3)$ and for $c > 0$ sufficiently small. In this range the critical point is found as a minimizer of F on $S(c)$.

Finally we prove

Theorem 3.1.15. *Let $p \in (\frac{10}{3}, 6)$. Any ground state of (3.1.4) is strongly unstable.*

Remark 3.1.16. The problem (3.1.4) is usually seen as the one of the *zero mass* type. Actually, in the *zero mass* case, there seems to be few results of stability or instability of standing waves. We are only aware of a stability result of M. Kaminaga and M. Ohta [68].

The chapter is organized as follows. In Section 3.2 we establish the mountain pass geometry of F on $S(c)$. In Section 3.3 we construct the special bounded Palais-Smale sequence at the level $\gamma(c)$. In Section 3.4 we show the convergence of the Palais-Smale sequence and we conclude the proof of Theorem 3.1.2. In Section 3.5 some parts of Theorem 3.1.4 are established. In Section 3.6 we prove Theorem 3.1.8 and Lemma 3.1.9. In Section 3.7 we prove Theorem 3.1.6 and using elements from [63] we end the proof of Theorem 3.1.4. Section 3.8 is devoted to the proof of Theorems 3.1.11, 3.1.12 and 3.1.15. Finally in Section 3.9 we discuss the nonlinear Schrödinger equation case.

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3.2 The mountain pass geometry on the constraint

In this section, we discuss the Mountain Pass Geometry ("MP Geometry" for short) of the functional F on the L^2-constraint $S(c)$. We show the following:

Theorem 3.2.1. *When $p \in (\frac{10}{3}, 6)$, for any $c > 0$, F has a MP geometry on the constraint $S(c)$.*

Before proving Theorem 3.2.1 we establish some lemmas. We first introduce the Cazenave scaling [37]. For $u \in S(c)$, we set $u^t(x) = t^{\frac{3}{2}}u(tx), t > 0$, then

$$\begin{aligned} A(u^t) &= t^2 A(u) \quad , \quad D(u^t) = D(u), \\ B(u^t) &= tB(u) \quad , \quad C(u^t) = t^{\frac{3}{2}(p-2)}C(u). \end{aligned}$$

Thus

$$F(u^t) = \frac{t^2}{2}A(u) + \frac{t}{4}B(u) - \frac{t^{\frac{3}{2}(p-2)}}{p}C(u). \tag{3.2.1}$$

Lemma 3.2.2. *Let $u \in S(c)$, $c > 0$ be arbitrary but fixed and $p \in (\frac{10}{3}, 6)$, then:*

(1) $A(u^t) \to \infty$ and $F(u^t) \to -\infty$, as $t \to \infty$.

(2) There exists $k_0 > 0$ such that $Q(u) > 0$ if $||\nabla u||_2 \leq k_0$ and $C(u) \geq k_0$ if $Q(u) = 0$.

(3) If $F(u) < 0$ then $Q(u) < 0$.

Proof. We notice that

$$F(u) - \frac{2}{3(p-2)}Q(u) = \frac{3p-10}{6(p-2)}A(u) + \frac{3p-8}{12(p-2)}B(u). \tag{3.2.2}$$

Thus (3) holds since the RHS is always positive. Moreover, thanks to Gagliardo-Nirenberg's inequality there exists a constant $C > 0$ such that

$$Q(u) \geq A(u) - C \cdot A(u)^{\frac{3(p-2)}{4}} D(u)^{\frac{6-p}{4}}.$$

The fact that $\frac{3(p-2)}{4} > 1$ insures that $Q(u) > 0$ for sufficiently small $A(u)$. Also when $Q(u) = 0$

$$C(u) = \frac{2p}{3(p-2)}\Big[A(u) + \frac{1}{4}B(u)\Big] \geq \frac{2p}{3(p-2)}A(u)$$

and this ends the proof of (2). Finally (1) follows directly from (3.2.1) and since $A(u^t) = t^2 A(u)$. □

Our next lemma is inspired by [37, Lemma 8.2.5].

Lemma 3.2.3. *When $p \in (\frac{10}{3}, 6)$, given $u \in S(c)$ we have:*

(1) There exists a unique $t^\star(u) > 0$, such that $u^{t^\star} \in V(c)$;

(2) The mapping $t \mapsto F(u^t)$ is concave on $[t^\star, \infty)$;

(3) $t^\star(u) < 1$ if and only if $Q(u) < 0$;

(4) $t^\star(u) = 1$ if and only if $Q(u) = 0$;

(5) $Q(u^t)\begin{cases} > 0, \ \forall\, t \in (0,\ t^\star(u)); \\ < 0, \ \forall\, t \in (t^\star(u), \infty). \end{cases}$

(6) $F(u^t) < F(u^{t^\star})$, for any $t > 0$ and $t \neq t^\star$;

(7) $\frac{\partial}{\partial t}F(u^t) = \frac{1}{t}Q(u^t)$, $\forall t > 0$.

Proof. Since

$$F(u^t) = \frac{t^2}{2}A(u) + \frac{t}{4}B(u) - \frac{t^{\frac{3}{2}(p-2)}}{p}C(u),$$

we have that

$$\frac{\partial}{\partial t}F(u^t) = tA(u) + \frac{1}{4}B(u) - \frac{3(p-2)}{2p}t^{\frac{3}{2}(p-2)-1}C(u) = \frac{1}{t}Q(u^t),$$

and this proves (7). Now we denote

$$y(t) = tA(u) + \frac{1}{4}B(u) - \frac{3(p-2)}{2p}t^{\frac{3}{2}(p-2)-1}C(u),$$

and observe that $Q(u^t) = t \cdot y(t)$. After direct calculations, we see that:

$$\begin{aligned} y'(t) &= A(u) - \frac{3(p-2)(3p-8)}{4p}t^{\frac{3p-10}{2}}C(u); \\ y''(y) &= -\frac{3(p-2)(3p-8)}{4p} \cdot \frac{3p-10}{2} \cdot t^{\frac{3p-12}{2}}C(u). \end{aligned}$$

From the expression of $y'(t)$ we know that $y'(t)$ has a unique zero that we denote $t_0 > 0$. Since $p \in (\frac{10}{3}, 6)$ we see that $y''(t) < 0$ and t_0 is the unique maximum point of $y(t)$. Thus in particular the function $y(t)$ satisfies:

(i) $y(t_0) = \max_{t>0} y(t)$;

(ii) $y(0) = \frac{1}{4}B(u)$;

(iii) $\lim_{t\to\infty} y(t) = -\infty$;

(iv) $y(t)$ decreases strictly in $[t_0, +\infty)$ and increases strictly in $(0, t_0]$.

Since $B(u) \neq 0$, by the continuity of $y(t)$, we deduce that $y(t)$ has a unique zero $t^\star > 0$. Then $Q(u^{t^\star}) = 0$ and point (1) follows. Point (2) (3) and (5) are also easy consequences of (i)-(iv). Since $\frac{\partial}{\partial t}F(u^t)|_{t=t^\star} = 0$, $\frac{\partial^2}{\partial t^2}F(u^t)|_{t=t^\star} = y'(t^\star) < 0$ and $t^\star$ is unique we get (4) and (6). □

Remark 3.2.4. It is not difficult to observe from the proof that Lemma 3.2.3 holds also for $p = \frac{10}{3}$.

Proof of Theorem 3.2.1. We denote

$$\alpha_k := \sup_{u\in C_k} F(u) \quad \text{and} \quad \beta_k := \inf_{u\in C_k} F(u)$$

where

$$C_k := \{u \in S(c) : A(u) = k, k > 0\}.$$

Let us show that there exist $0 < k_1 < k_2$ such that

$$\alpha_k < \beta_{k_2} \text{ for all } k \in (0, k_1] \text{ and } Q(u) > 0 \text{ if } A(u) < k_2. \tag{3.2.3}$$

Notice that, from the Hardy-Littlewood-Sobolev and Gagliardo-Nirenberg inequalities, it follows that

$$\begin{aligned} F(u) &\leq \frac{1}{2}A(u) + \frac{1}{4}B(u) \leq \frac{1}{2}A(u) + C \cdot \|u\|^4_{L^{\frac{12}{5}}} \\ &\leq \frac{1}{2}A(u) + C \cdot A(u)^{\frac{1}{2}} \cdot D(u)^{\frac{3}{2}}. \end{aligned}$$

In particular $\alpha_{k_1} \to 0^+$ as $k_1 \to 0^+$. On the other hand still by the Gagliardo-Nirenberg inequality we have

$$F(u) \geq \frac{1}{2}A(u) - \frac{1}{p}C(u) \geq \frac{1}{2}A(u) - C \cdot A(u)^{\frac{3(p-2)}{4}} \cdot D(u)^{\frac{6-p}{4}}.$$

Thus, since $\frac{3(p-2)}{4} > 1$, $\beta_{k_2} \geq \frac{1}{4}k_2$ for any $k_2 > 0$ small enough. These two observations and Lemma 3.2.2 (2) prove that (3.2.3) hold. We now fix a $k_1 > 0$ and a $k_2 > 0$ as in (3.2.3). Thus for

$$\Gamma_c = \{g \in C([0,1], S(c)),\ g(0) \in A_{k_1}, F(g(1)) < 0\},$$

if $\Gamma_c \neq \emptyset$, then from the definition of $\gamma(c)$, we have $\gamma(c) \geq \beta_{k_2} > 0$. Thus we only need to verify that $\Gamma_c \neq \emptyset$. But this fact follows from Lemma 3.2.2 (1). At his point, we are done with the proof. $\square$

Remark 3.2.5. As it is clear from the proof of Theorem 3.2.1 we can assume without restriction that

$$\sup_{u \in A_{K_c}} F(u) < \gamma(c)/2$$

where A_{K_c} is introduced in the Definition 3.1.1.

Lemma 3.2.6. *When $p \in (\frac{10}{3}, 6)$, we have*

$$\gamma(c) = \inf_{u \in V(c)} F(u).$$

Proof. Let us argue by contradiction. Suppose there exists $v \in V(c)$ such that $F(v) < \gamma(c)$, and let, for $\lambda > 0$,

$$v^{\lambda}(x) = \lambda^{3/2} v(\lambda x).$$

Then, since $A(v^{\lambda}) = \lambda^2 A(v)$ there exists $0 < \lambda_1 < 1$ sufficiently small so that $v^{\lambda_1} \in A_{k_1}$. Also by Lemma 3.2.2 (1) there exists a $\lambda_2 > 1$ sufficiently large so that $F(v^{\lambda_2}) < 0$. Therefore if we define

$$g(t) = v^{(1-t)\lambda_1 + t\lambda_2}, \quad \text{for } t \in [0,1]$$

we obtain a path in Γ_c. By definition of $\gamma(c)$ and using Lemma 3.2.3,

$$\gamma(c) \leq \max_{t \in [0,1]} F(g(t)) = F\Big(g(\frac{1-\lambda_1}{\lambda_2 - \lambda_1})\Big) = F(v),$$

and thus

$$\gamma(c) \leq \inf_{u \in V(c)} F(u).$$

On other hand thanks to Lemma 3.2.2 any path in Γ_c crosses $V(c)$ and hence

$$\max_{t \in [0,1]} F(g(t)) \geq \inf_{u \in V(c)} F(u).$$

$\square$

3.3 Localization of a PS sequence

In this section we prove a localization lemma for a specific Palais-Smale sequence $\{u_n\} \subset S(c)$ for F constrained to $S(c)$. From this localization we deduce that the sequence is bounded and that $Q(u_n) = o(1)$. This last property will be essential later to establish the compactness of the sequence. First we observe that, for any fixed $c > 0$, the set

$$L := \Big\{u \in V(c), F(u) \leq \gamma(c) + 1\Big\}$$

is bounded. This follows directly from the observation that

$$F(u) - \frac{2}{3(p-2)}Q(u) = \frac{3p-10}{6(p-2)}A(u) + \frac{3p-8}{12(p-2)}B(u) \tag{3.3.1}$$

and the fact that $\frac{3p-10}{6(p-2)} > 0$, $\frac{3p-8}{12(p-2)} > 0$ if $p \in (\frac{10}{3}, 6)$.

Let $R_0 > 0$ be such that $L \subset B(0, R_0)$ where $B(0, R_0) := \{u \in H^1(\mathbb{R}^3), ||u|| \leq R_0\}$.

The crucial localization result is the following.

Lemma 3.3.1. *Let $p \in (\frac{10}{3}, 6)$ and*

$$K_\mu := \Big\{u \in S(c) \ s.t. \ |F(u) - \gamma(c)| \leq \mu, \ dist(u, V(c)) \leq 2\mu, \ ||F^{'}|_{S(c)}(u)||_{H^{-1}} \leq 2\mu\Big\},$$

then for any $\mu > 0$, the set $K_\mu \cap B(0, 3R_0)$ is not empty.

In order to prove Lemma 3.3.1 we need to develop a deformation argument on $S(c)$. Following [20] we recall that, for any $c > 0$, $S(c)$ is a submanifold of $H^1(\mathbb{R}^3)$ with codimension 1 and the tangent space at a point $\bar{u} \in S(c)$ is defined as

$$T_{\bar{u}}S(c) := \Big\{v \in H^1(\mathbb{R}^3) \text{ s.t. } (\bar{u}, v)_2 = 0\Big\}.$$

The restriction $F_{|_{S(c)}} : S(c) \to \mathbb{R}$ is a C^1 functional on $S(c)$ and for any $\bar{u} \in S(c)$ and any $v \in T_{\bar{u}}S(c)$

$$\langle F'_{|_{S(c)}}(\bar{u}), v\rangle = \langle F'(\bar{u}), v\rangle.$$

We use the notation $||dF_{|_{S(c)}}(\bar{u})||$ to indicate the norm in the cotangent space $T_{\bar{u}}S(c)'$, i.e the dual norm induced by the norm of $T_{\bar{u}}S(c)$, i.e

$$||dF_{|_{S(c)}}(\bar{u})|| := \sup_{||v|| \leq 1, \ v \in T_{\bar{u}}S(c)} |\langle dF(\bar{u}), v\rangle|.$$

Let $\tilde{S}(c) := \{u \in S(c) \text{ s.t. } dF|_{S(c)}(u) \neq 0\}$. We know from [20] that there exists a locally Lipschitz pseudo gradient vector field $Y \in \mathcal{C}^1(\tilde{S}(c), T(S(c))$ (here $T(S(c))$ is the tangent bundle) such that

$$\|Y(u)\| \leq 2\, ||dF_{|_{S(c)}}(u)||, \tag{3.3.2}$$

and

$$\langle F'_{|_{S(c)}}(\bar{u}), Y(u)\rangle \geq ||dF_{|_{S(c)}}(u)||^2, \tag{3.3.3}$$

for any $u \in \tilde{S}(c)$. Note that $||Y(u)|| \neq 0$ for $u \in \tilde{S}(c)$ thanks to (3.3.3). Now for an arbitrary but fixed $\mu > 0$ we consider the sets

$$\tilde{N}_\mu := \{u \in S(c) \text{ s.t. } |F(u) - \gamma(c)| \leq \mu, \ dist(u, V(c)) \leq 2\mu, \ ||Y(u)|| \geq 2\mu\}$$

$$N_\mu := \{u \in S(c) \text{ s.t. } |F(u) - \gamma(c)| < 2\mu\}$$

where, for a subset $\mathcal{A}$ of $S(c)$, $dist(x, \mathcal{A}) := \inf\{||x - y|| : y \in \mathcal{A}\}$. Assuming that $\tilde{N}_\mu$ is non empty there exists a locally Lipschitz function $g : S(c) \to [0, 1]$ such that

$$g = \begin{cases} 1 & \text{on } \tilde{N}_\mu, \\ 0 & \text{on } N_\mu^c. \end{cases}$$

We also define on $S(c)$ the vector field W by

$$W(u) := \begin{cases} -g(u)\frac{Y(u)}{||Y(u)||}, & \text{if } u \in \tilde{S}(c), \\ 0, & \text{if } u \in S(c)\backslash\tilde{S}(c). \end{cases} \tag{3.3.4}$$

and the pseudo gradient flow

$$\begin{cases} \frac{d}{dt}\eta(t,u) = W(\eta(t,u)), \\ \eta(0,u) = u. \end{cases} \tag{3.3.5}$$

The existence of a unique solution $\eta(t,\cdot)$ of (3.3.5) defined for all $t \in \mathbb{R}$ follows from standard arguments and we refer to [20, Lemma 5] for this. Let us recall some of its basic properties that will be useful to us

- $\eta(t,\cdot)$ is a homeomorphism of $S(c)$;
- $\eta(t,u) = u$ for all $t \in \mathbb{R}$ if $|F(u) - \gamma(c)| \geq 2\mu$;
- $\frac{d}{dt}F(\eta(t,u)) = \langle dF(\eta(t,u)), W(\eta(t,u))\rangle \leq 0$ for all $t \in \mathbb{R}$ and $u \in S(c)$.

Proof of Lemma 3.3.1. : Let us define, for $\mu > 0$,

$$\Lambda_\mu := \Big\{u \in S(c) \text{ s.t. } |F(u) - \gamma(c)| \leq \mu,\ dist(u, V(c)) \leq 2\mu\Big\}.$$

In order to prove Lemma 3.3.1 we argue by contradiction assuming that there exists $\bar{\mu} \in (0, \gamma(c)/4)$ such that

$$u \in \Lambda_{\bar{\mu}} \cap B(0, 3R_0) \implies ||F^{'}|_{S(c)}(u)||_{H^{-1}} > 2\bar{\mu}. \tag{3.3.6}$$

Then it follows from (3.3.3) that

$$u \in \Lambda_{\bar{\mu}} \cap B(0, 3R_0) \implies u \in \tilde{N}_{\bar{\mu}}. \tag{3.3.7}$$

Also notice that, since by (3.3.5),

$$||\frac{d}{dt}\eta(t,u)|| \leq 1, \quad \forall t \geq 0, \forall u \in S(c),$$

there exists $s_0 > 0$ depending on $\bar{\mu} > 0$ such that, for all $s \in (0, s_0)$,

$$u \in \Lambda_{\frac{\bar{\mu}}{2}} \cap B(0, 2R_0) \implies \eta(s,u) \in B(0, 3R_0) \text{ and } dist(\eta(s,u), V(c)) \leq 2\bar{\mu}. \tag{3.3.8}$$

We claim that, taking $\varepsilon > 0$ sufficiently small, we can construct a path $g_\varepsilon(t) \in \Gamma_c$ such that

$$\max_{t\in[0,1]} F(g_\varepsilon(t)) \leq \gamma(c) + \varepsilon$$

and

$$F(g_\varepsilon(t)) \geq \gamma(c) \implies g_\varepsilon(t) \in \Lambda_{\frac{\bar{\mu}}{2}} \cap B(0, 2R_0). \tag{3.3.9}$$

Indeed, for $\varepsilon > 0$ small, let $u \in V(c)$ be such that $F(u) \leq \gamma(c) + \varepsilon$ and consider the path defined in Lemma 3.2.6 by

$$g_\varepsilon(t) = u^{(1-t)\lambda_1 + t\lambda_2}, \quad \text{for } t \in [0,1]. \tag{3.3.10}$$

Clearly

$$\max_{t\in[0,1]} F(g_\varepsilon(t)) \leq \gamma(c) + \varepsilon.$$

Also for $t_\varepsilon^* > 0$ such that $(1 - t_\varepsilon^*)\lambda_1 + t_\varepsilon^*\lambda_2 = 1$ we have, since $g_\varepsilon(t_\varepsilon^*) \in V(c)$, that

$$\frac{d^2}{d^2s} F(g_\varepsilon(s))_{|t_\varepsilon^*} = -\frac{1}{4}B(u) - \frac{3}{2p}(p-2)(5 - \frac{3}{2}p)C(u) \leq -C \cdot k_0 < 0 \tag{3.3.11}$$

where $k_0 > 0$ is given in Lemma 3.2.2 (2). The estimate (3.3.11) is uniform with respect to the choice of $\varepsilon > 0$ and of $u \in V(c)$. Thus, by Taylor's formula, it is readily seen that

$$\{t \in [0,1] : F(g_\varepsilon(t)) \geq \gamma(c)\} \subset [t_\varepsilon^* - \alpha_\varepsilon, t_\varepsilon^* + \alpha_\varepsilon]$$

for some $\alpha_\varepsilon > 0$ with $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. The claim (3.3.10) follows for continuity arguments.

We fix a $\varepsilon \in (0, \frac{1}{4}\bar{\mu}s_0)$ such that (3.3.9) hold. Applying the pseudo gradient flow, constructed with $\bar{\mu} > 0$, on $g_\varepsilon(t)$ we see that $\eta(s, g_\varepsilon(\cdot)) \in \Gamma_c$ for all $s > 0$. Indeed $\eta(s,u) = u$ for all $s > 0$ if $|F(u) - \gamma(c)| \geq 2\bar{\mu}$ and we conclude by Remark 3.2.5.

We claim that taking $s^* := \dfrac{4\varepsilon}{\bar{\mu}} < s_0$

$$\max_{t\in[0,1]} F(\eta(s^*, g_\varepsilon(t))) < \gamma(c). \tag{3.3.12}$$

If (3.3.12) holds we have a contradiction with the definition of $\gamma(c)$ and thus the lemma is proved. To prove (3.3.12) for simplicity we set $w = g_\varepsilon(t)$ where $t \in [0,1]$. If $F(w) < \gamma(c)$ there is nothing to prove since then $F(\eta(s^*, w) \leq F(w) < \gamma(c)$ for any $s > 0$. If $F(w) \geq \gamma(c)$ we assume by contradiction that $F(\eta(s,w)) \geq \gamma(c)$ for all $s \in [0, s^*]$. Then by (3.3.8) and (3.3.9), $\eta(s,w) \in \Lambda_{\bar{\mu}} \cap B(0, 3R_0)$ for all $s \in [0, s^*]$. In particular $||Y(\eta(s,w))|| \geq 2\bar{\mu}$ and $g(\eta(s,w)) = 1$ for all $s \in [0, s^*]$. Thus

$$\frac{d}{ds} F(\eta(s,w)) = \langle dF(\eta(s,w)), -\frac{Y(\eta(t,u))}{||Y(\eta(t,u))||}\rangle.$$

By integration, and since $s^* = \dfrac{4\varepsilon}{\bar{\mu}}$, we get

$$F(\eta(s^*, w)) \leq F(w) - \bar{\mu}s^* \leq (\gamma(c) + \varepsilon) - 2\varepsilon < \gamma(c) - \varepsilon.$$

This proves the claim (3.3.12) and the lemma. □

Lemma 3.3.2. *Let $p \in (\frac{10}{3}, 6)$, then there exists a sequence $\{u_n\} \subset S(c)$ and a constant $\alpha > 0$ fulfilling*

$$Q(u_n) = o(1), \quad F(u_n) = \gamma(c) + o(1),$$
$$||F'|_{S(c)}(u_n)||_{H^{-1}} = o(1), \ ||u_n|| \leq \alpha.$$

Proof. First let us consider $\{u_n\} \subset S(c)$ such that $\{u_n\} \subset B(0, 3R_0)$,

$$dist(u_n, V(c)) = o(1), \ |F(u_n) - \gamma(c)| = o(1), \ ||F'|_{S(c)}(u_n)||_{H^{-1}} = o(1).$$

Such sequence exists thanks to Lemma 3.3.1. To prove the lemma we just have to show that $Q(u_n) = o(1)$. It is readily checked that $||dQ(\cdot)||_{H^{-1}}$ is bounded on any bounded set of $H^1(\mathbb{R}^3)$ and thus in particular on $B(0,3R_0)$. Now, for any $n \in \mathbb{N}$ and any $w \in V(c)$ we can write

$$Q(u_n) = Q(w) + dQ(au_n + (1-a)w)(u_n - w)$$

where $a \in [0,1]$. Thus since $Q(w) = 0$ we have

$$|Q(u_n)| \leq \max_{u \in B(0,3R_0)} ||dQ||_{H^{-1}} ||u_n - w||. \tag{3.3.13}$$

Finally choosing $\{w_m\} \subset V(c)$ such that

$$||u_n - w_m|| \to dist(u_n, V(c)) \text{ as } m \to \infty,$$

since $dist(u_n, V(c)) \to 0$ we obtain from (3.3.13) that $Q(u_n) = o(1)$. $\square$

3.4 Compactness of our Palais-Smale sequence

Proposition 3.4.1. *Let $\{v_n\} \subset S(c)$ be a bounded Palais-Smale for F restricted to $S(c)$ such that $F(v_n) \to \gamma(c)$. Then there is a sequence $\{\lambda_n\} \subset \mathbb{R}$, such that, up to a subsequence:*

(1) $v_n \rightharpoonup v_c$ weakly in $H^1(\mathbb{R}^3)$;

(2) $\lambda_n \to \lambda_c$ in $\mathbb{R}$;

(3) $-\Delta v_n - \lambda_n v_n + (|x|^{-1} * |v_n|^2)v_n - |v_n|^{p-2}v_n \to 0$ in $H^{-1}(\mathbb{R}^3)$;

(4) $-\Delta v_n - \lambda_c v_n + (|x|^{-1} * |v_n|^2)v_n - |v_n|^{p-2}v_n \to 0$ in $H^{-1}(\mathbb{R}^3)$;

(5) $-\Delta v_c - \lambda_c v_c + (|x|^{-1} * |v_c|^2)v_c - |v_c|^{p-2}v_c = 0$ in $H^{-1}(\mathbb{R}^3)$.

Proof. Point (1) is trivial. Since $\{v_n\} \subset H^1(\mathbb{R}^3)$ is bounded, following Berestycki and Lions [20, Lemma 3], we know that:

$$\begin{aligned} F'|_{S(c)}(v_n) &\longrightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3) \\ &\iff F'(v_n) - \langle F'(v_n), v_n \rangle v_n \longrightarrow 0 \text{ in } H^{-1}(\mathbb{R}^3). \end{aligned}$$

Thus, for any $w \in H^1(\mathbb{R}^3)$,

$$\begin{aligned} \langle F'(v_n) - \langle F'(v_n), v_n \rangle v_n, w \rangle = &\int_{\mathbb{R}^3} \nabla v_n \nabla w dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_n(x)|^2}{|x-y|} v_n(y) w(y) dx dy \\ &- \int_{\mathbb{R}^3} |v_n|^{p-2} v_n w dx - \lambda_n \int_{\mathbb{R}^3} v_n(x) w(x) dx, \end{aligned}$$

with

$$\lambda_n := \frac{1}{\|v_n\|_2^2} \Big\{ \|\nabla v_n\|_2^2 + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v_n(x)|^2 v_n(x)^2}{|x-y|} dx dy - \|v_n\|_p^p \Big\}. \tag{3.4.1}$$

Thus we obtain (3) with $\{\lambda_n\} \subset \mathbb{R}$ defined by (3.4.1). If (2) holds then (4) follows immediately from (3). To prove (2), it is enough to verify that $\{\lambda_n\} \subset \mathbb{R}$ is bounded. But since $\{v_n\} \subset H^1(\mathbb{R}^N)$ is bounded, by the Hardy-Littlewood-Sobolev and Gagliardo-Nirenberg inequalities, it is easy to see that all terms in the RHS of (3.4.1) are bounded. Finally we refer to [112, Lemma 2.2] for a proof of (5). $\square$

Lemma 3.4.2. *Let $p \in (\frac{10}{3}, 6)$ and $\{u_n\} \subset S(c)$ be a bounded sequence such that*

$$Q(u_n) = o(1) \quad \text{and} \quad F(u_n) \to \gamma(c) \text{ with } \gamma(c) > 0,$$

then, up to a subsequence and up to translation $u_n \rightharpoonup \bar{u} \neq 0$.

Proof. If the lemma does not hold it means by standard arguments that $\{u_n\} \subset S(c)$ is vanishing and thus that $C(u_n) = o(1)$ (see [83]). Thus let us argue by contradiction assuming that $C(u_n) = o(1)$, i.e. that, since $Q(u_n) = o(1)$, $A(u_n) + \frac{1}{4}B(u_n) = o(1)$. Now from (3.3.1) we immediately deduce that $F(u_n) = o(1)$ and this contradicts the assumption that $F(u_n) \to \gamma(c) > 0$. □

Lemma 3.4.3. *Let $p \in (\frac{10}{3}, 6)$, $\lambda \in \mathbb{R}$. If $v \in H^1(\mathbb{R}^3)$ is a weak solution of*

$$-\Delta v + \left(|x|^{-1} * |v|^2\right) v - |v|^{p-2} v = \lambda v \tag{3.4.2}$$

then $Q(v) = 0$. Moreover if $\lambda \geq 0$, there exists a constant $c_0 > 0$ independent on $\lambda \in \mathbb{R}$ such that the only solution of (3.4.2) fulfilling $||v||_2^2 \leq c_0$ is the null function.

Proof. The following Pohozaev type identity holds for $v \in H^1(\mathbb{R}^3)$ weak solution of (3.4.2), see [43],

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{5}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dxdy - \frac{3}{p}\int_{\mathbb{R}^3} |v|^p \, dx = \frac{3\lambda}{2}\int_{\mathbb{R}^3} |v|^2 \, dx.$$

By multiplying (3.4.2) by v and integrating we derive a second identity

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v(x)|^2 |v(y)|^2}{|x-y|} dxdy - \int_{\mathbb{R}^3} |v|^p \, dx = \lambda \int_{\mathbb{R}^3} |v|^2 \, dx.$$

With simple calculus we obtain the following relations

$$\begin{aligned} &A(v) + \frac{1}{4}B(v) - 3\left(\frac{p-2}{2p}\right) C(v) = 0, \\ &(\frac{p-6}{3p-6})A(v) + (\frac{5p-12}{3p-6})\frac{B(v)}{2} = \lambda D(v). \end{aligned} \tag{3.4.3}$$

The first relation of (3.4.3) is $Q(v) = 0$. This identity together with the Gagliardo-Nirenberg inequality assures the existence of a constant $C > 0$ such that

$$A(v) - C \cdot A(v)^{\frac{3(p-2)}{4}} D(v)^{\frac{6-p}{4}} \leq A(v) - 3\left(\frac{p-2}{2p}\right) C(v) \leq 0, \tag{3.4.4}$$

i.e

$$A(v)^{\frac{10-3p}{4}} \leq C \cdot D(v)^{\frac{6-p}{4}}. \tag{3.4.5}$$

Now we recall that by the Hardy-Littlehood-Sobolev and Gagliardo-Nirenberg inequalities we have

$$B(v) \leq C \cdot A(v)^{\frac{1}{2}} D(v)^{\frac{3}{2}}, \tag{3.4.6}$$

then, from the second relation of (3.4.3) we obtain

$$\lambda D(v) \leq (\frac{p-6}{3p-6})A(v) + C \cdot A(v)^{\frac{1}{2}} D(v)^{\frac{3}{2}}. \tag{3.4.7}$$

Notice that (3.4.5) tells us that, for any solution u of (3.4.2) with small L^2-norm, $A(u)$ must be large. This fact assures that the left hand side of (3.4.7) cannot be non negative when $D(v)$ is sufficiently small. □

As a consequence of the proof of Lemma 3.4.3, we could immediately prove Lemma 3.1.3.

Proof of Lemma 3.1.3. Let (v_c, λ_c) solves weakly (3.4.2). Then from (3.4.5) we have that $A(v_c) \to +\infty$ as $c \to 0$. Thus it follows from (3.4.7) that

$$\lambda_c \to -\infty, \quad \text{as } c \to 0.$$

This yields that $\lambda_c \to -\infty$ as $c \to 0$. □

Lemma 3.4.4. *Let $p \in (\frac{10}{3}, 6)$. Assume that the bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ given by Lemma 3.3.2 is weakly convergent, up to translations, to the nonzero function $\bar{u}$. Moreover assume that*

$$\forall c_1 \in (0, c), \quad \gamma(c_1) > \gamma(c). \tag{3.4.8}$$

Then $\|u_n - \bar{u}\| \to 0$. In particular it follows that $\bar{u} \in S(c)$ and $F(\bar{u}) = \gamma(c)$.

Proof. Let $T(u) := \frac{1}{4}B(u) - \frac{1}{p}C(u)$ such that

$$F(u) = \frac{1}{2}\|\nabla u\|_2^2 + T(u). \tag{3.4.9}$$

In [14] or [112] it is shown that the nonlinear term T fulfills the following splitting properties of Brezis-Lieb type (see [24]),

$$T(u_n - \bar{u}) + T(\bar{u}) = T(u_n) + o(1). \tag{3.4.10}$$

We argue by contradiction and assume that $c_1 = ||\bar{u}||_2^2 < c$. By Proposition 3.4.1 (5) and Lemma 3.4.3 we have $Q(\bar{u}) = 0$ and thus $\bar{u} \in V(c_1)$. Now since $u_n - \bar{u} \rightharpoonup 0$,

$$\|\nabla(u_n - \bar{u})\|_2^2 + \|\nabla \bar{u}\|_2^2 = \|\nabla u_n\|_2^2 + o(1). \tag{3.4.11}$$

Also since $\{u_n\} \subset S(c)$ is a sequence at the level $\gamma(c)$ we get

$$\frac{1}{2}\|\nabla u_n\|_2^2 + T(u_n) = \gamma(c) + o(1). \tag{3.4.12}$$

Combining (3.4.10) - (3.4.12) we deduce that

$$\frac{1}{2}\|\nabla(u_n - \bar{u})\|_2^2 + \frac{1}{2}\|\nabla \bar{u}\|_2^2 + T(u_n - \bar{u}) + T(\bar{u}) = \gamma(c) + o(1). \tag{3.4.13}$$

At this point, using that $\bar{u} \in V(c_1)$ and Lemma 3.2.6 we get from (3.4.13) that

$$F(u_n - \bar{u}) + \gamma(c_1) \leq \gamma(c) + o(1). \tag{3.4.14}$$

On the other hand,

$$F(u_n - \bar{u}) - \frac{2}{3(p-2)}Q(u_n - \bar{u}) = \frac{3p-10}{6(p-2)}A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)}B(u_n - \bar{u}) \tag{3.4.15}$$

and

$$Q(u_n - \bar{u}) = Q(u_n - \bar{u}) + Q(\bar{u}) = Q(u_n) + o(1) = o(1). \tag{3.4.16}$$

From (3.4.15) and (3.2.2) we deduce that $F(u_n - \bar{u}) \geq o(1)$. But then from (3.4.14) we obtain a contradiction with (3.4.8). This contradiction proves that $\|\bar{u}\|_2^2 = c$ and $F(\bar{u}) \geq \gamma(c)$. Now still by (3.4.14) we get $F(u_n - \bar{u}) \leq o(1)$ and thanks to (3.4.15) and (3.4.16) $A(u_n - \bar{u}) = o(1)$. i.e $\|\nabla(u_n - \bar{u})\|_2 = o(1)$. □

Lemma 3.4.5. *Let $p \in (\frac{10}{3}, 6)$. Assume that the bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ given by Lemma 3.3.2 is weakly convergent, up to translations, to the nonzero function $\bar{u}$. Moreover assume that*

$$\forall c_1 \in (0, c),\ \gamma(c_1) \geq \gamma(c) \tag{3.4.17}$$

and that the Lagrange multiplier given by Proposition 3.4.1 fulfills

$$\lambda_c \neq 0.$$

Then $\|u_n - \bar{u}\| \to 0$. In particular it follows that $\bar{u} \in S(c)$ and $F(\bar{u}) = \gamma(c)$.

Proof. Let us argue as in Lemma 3.4.4. We obtain again

$$F((u_n - \bar{u})) + \gamma(c_1) \leq \gamma(c) + o(1),$$

$$F(u_n - \bar{u}) - \frac{2}{3(p-2)}Q(u_n - \bar{u}) = \frac{3p-10}{6(p-2)}A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)}B(u_n - \bar{u})$$

and

$$Q(u_n - \bar{u}) = Q(u_n - \bar{u}) + Q(\bar{u}) = Q(u_n) + o(1) = o(1).$$

Thanks to (3.4.17) we conclude that

$$\frac{3p-10}{6(p-2)}A(u_n - \bar{u}) + \frac{3p-8}{12(p-2)}B(u_n - \bar{u}) = o(1).$$

Then

$$A(u_n - \bar{u}) = o(1), B(u_n - \bar{u}) = o(1) \text{ and also } C(u_n - \bar{u}) = o(1), \tag{3.4.18}$$

since $Q(u_n - \bar{u}) = o(1)$. Now we use (5) of Proposition 3.4.1, i.e

$$A(u_n) - \lambda_c D(u_n) + B(u_n) - C(u_n) = A(\bar{u}) - \lambda_c D(\bar{u}) + B(\bar{u}) - C(\bar{u}) + o(1).$$

Thanks to the splitting properties of $A(u), B(u), C(u)$ and to (3.4.18) we get

$$-\lambda_c D(u_n) = -\lambda_c D(\bar{u}) + o(1),$$

which implies $D(u_n - \bar{u}) = o(1)$, i.e $||u_n - \bar{u}||_2 = o(1)$. From this point we conclude as in the proof of Lemma 3.4.4. ☐

Admitting for the moment that $c \to \gamma(c)$ is non-increasing (we shall prove it in the next section) we can now complete the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. By Lemmas 3.3.2 and 3.4.2 there exists a bounded Palais-Smale sequence $\{u_n\} \subset S(c)$ such that, up to translation, $u_n \rightharpoonup u_c \neq 0$. Thus, by Proposition 3.4.1 there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3)\backslash\{0\}\times\mathbb{R}$ solves (E_λ). Now by Lemma 3.4.3 there exists a $c_0 > 0$ such that $\lambda_c < 0$ if $c \in (0, c_0)$. Also we know from Theorem 3.1.4 (ii) that (3.4.17) holds. At this point the proof follows from Lemma 3.4.5. ☐

3.5 The behavior of $c \to \gamma(c)$

In this section we give the proof of Theorem 3.1.4. Let us denote

$$\gamma_1(c) := \inf_{u \in S(c)} \max_{t>0} F(u^t), \tag{3.5.1}$$

and

$$\gamma_2(c) := \inf_{u \in V(c)} F(u). \tag{3.5.2}$$

Lemma 3.5.1. *For $p \in (\frac{10}{3}, 6)$, we have:*

$$\gamma(c) = \gamma_1(c) = \gamma_2(c).$$

Proof. When $p \in (\frac{10}{3}, 6)$, from Lemma 3.2.6, we know that $\gamma(c) = \gamma_2(c)$. In addition, by Lemma 3.2.3, it is clear that for any $u \in S(c)$, there exists a unique $t_0 > 0$, such that $u^{t_0} \in V(c)$ and $\max_{t>0} F(u^t) = F(u^{t_0}) \geq \gamma_2(c)$, thus we get $\gamma_1(c) \geq \gamma_2(c)$. Meanwhile, for any $u \in V(c)$, $\max_{t>0} F(u_t) = F(u)$ and this readily implies that $\gamma_1(c) \leq \gamma_2(c)$. Thus we conclude that $\gamma_1(c) = \gamma_2(c)$. □

Lemma 3.5.2. *Denote*

$$f(a, b, c) := \max_{t>0} \left\{ a \cdot t^2 + b \cdot t - c \cdot t^{\frac{3}{2}(p-2)} \right\},$$

where $p \in (\frac{10}{3}, 6)$ and $a > 0, b \geq 0, c > 0$ which are totally independent of $t > 0$. Then the function: $(a, b, c) \mapsto f(a, b, c)$ is continuous in $\mathbb{R}^+ \times \mathbb{R}^c_- \times \mathbb{R}^+$ (here $\mathbb{R}^c_-$ denotes the set of non-negative real numbers).

Proof. Let $g(a, b, c, t) := a \cdot t^2 + b \cdot t - c \cdot t^{\frac{3}{2}(p-2)}$, then

$$\partial_t g(a, b, c, t) = 2a \cdot t + b - \frac{3}{2}(p-2) \cdot c \cdot t^{\frac{3p-8}{2}},$$

$$\partial^2_{tt} g(a, b, c, t) = 2a - \frac{3p-6}{2} \cdot \frac{3p-8}{2} \cdot c \cdot t^{\frac{3p-10}{2}}.$$

It's not difficult to see that for any (a_0, b_0, c_0) with $a_0 > 0, b_0 \geq 0, c_0 > 0$, there exists a unique $t_1 > 0$, such that $\partial_t g(a_0, b_0, c_0, t_1) = 0$ and $\partial^2_{tt} g(a_0, b_0, c_0, t_1) < 0$, thus $f(a_0, b_0, c_0) = g(a_0, b_0, c_0, t_1)$. Then applying the Implicit Function Theorem to the function $\partial_t g(a, b, c, t)$, we deduce the existence of a continuous function $t = t(a, b, c)$ in some neighborhood O of (a_0, b_0, c_0) that satisfies $\partial_t g(a, b, c, t(a, b, c)) = 0$, $\partial^2_{tt} g(a, b, c, t(a, b, c)) < 0$. Thus $f(a, b, c) = g(a, b, c, t(a, b, c))$ in O. Now since the function $g(a, b, c, t)$ is continuous in (a, b, c, t), it follows that $f(a, b, c)$ is continuous in (a_0, b_0, c_0). The point (a_0, b_0, c_0) being arbitrary this concludes the proof. □

Lemma 3.5.3. *Let $p \in (\frac{10}{3}, 6)$, then the function $c \to \gamma(c)$ is non increasing for $c > 0$.*

Proof. To show that $c \mapsto \gamma(c)$ is non increasing, it is enough to verify that: for any $c_1 < c_2$ and $\varepsilon > 0$ arbitrary, we have

$$\gamma(c_2) \leq \gamma(c_1) + \varepsilon. \tag{3.5.3}$$

By the definition of $\gamma_2(c_1)$, there exists $u_1 \in V(c_1)$ such that $F(u_1) \leq \gamma_2(c_1) + \frac{\varepsilon}{2}$. Thus by Lemma 3.5.1, we have

$$F(u_1) \leq \gamma(c_1) + \frac{\varepsilon}{2} \tag{3.5.4}$$

and also

$$F(u_1) = \max_{t>0} F(u_1^t). \tag{3.5.5}$$

We truncate u_1 into a function with compact support $\widetilde{u}_1$ as follows. Let $\eta \in C_0^\infty(\mathbb{R}^3)$ be radial and such that

$$\eta(x) = \begin{cases} 1, & |x| \le 1, \\ \in [0,1], & 1 < |x| < 2, \\ 0, & |x| \ge 2. \end{cases}$$

For any small $\delta > 0$, let

$$\widetilde{u}_1(x) = \eta(\delta x) \cdot u_1(x). \tag{3.5.6}$$

It is standard to show that $\widetilde{u}_1(x) \to u_1(x)$ in $H^1(\mathbb{R}^3)$ as $\delta \to 0$. Then, by continuity, we have, as $\delta \to 0$,

$$A(\widetilde{u}_1) \to A(u_1),\ B(\widetilde{u}_1) \to B(u_1) \text{ and } C(\widetilde{u}_1) \to C(u_1). \tag{3.5.7}$$

Thus applying Lemma 3.5.2, we deduce that there exists $\delta > 0$ small enough, such that

$$\begin{aligned} \max_{t>0} F(\widetilde{u}_1^t) &= \max_{t>0} \Big\{ \frac{t^2}{2} A(\widetilde{u}_1) + tB(\widetilde{u}_1) - t^{\frac{3}{2}(p-2)} C(\widetilde{u}_1) \Big\} \\ &\le \max_{t>0} \Big\{ \frac{t^2}{2} A(u_1) + tB(u_1) - t^{\frac{3}{2}(p-2)} C(u_1) \Big\} + \frac{\varepsilon}{4} \\ &= \max_{t>0} F(u_1^t) + \frac{\varepsilon}{4}. \end{aligned} \tag{3.5.8}$$

Now let $v(x) \in C_0^\infty(\mathbb{R}^3)$ be radial and such that *supp* $v \subset B_{2R_\delta+1} \backslash B_{2R_\delta}$. Here *supp* v denotes the support of v and $R_\delta = \frac{2}{\delta}$. Then we define

$$v_0 := v \cdot (c_2 - \|\widetilde{u}_1\|_2^2)/\|v\|_2^2,$$

for which we have $\|v_0\|_2^2 = c_2 - \|\widetilde{u}_1\|_2^2$. Finally letting $v_0^\lambda = \lambda^{\frac{3}{2}} v_0(\lambda x)$, for $\lambda \in (0,1)$, we have $\|v_0^\lambda\|_2^2 = \|v_0\|_2^2$ and

$$A(v_0^\lambda) = \lambda^2 \cdot A(v_0),\ B(v_0^\lambda) = \lambda \cdot B(v_0) \text{ and } C(v_0^\lambda) = \lambda^{\frac{3}{2}(p-2)} \cdot C(v_0). \tag{3.5.9}$$

Now for any $\lambda \in (0,1)$ we define $w_\lambda = \widetilde{u}_1 + v_0^\lambda$. We observe that

$$dist\{supp\ \widetilde{u}_1, supp\ v_0^\lambda\} \ge \frac{2R_\delta}{\lambda} - R_\delta = \frac{2}{\delta}(\frac{2}{\lambda} - 1). \tag{3.5.10}$$

Thus $\|w_\lambda\|_2^2 = \|\widetilde{u}_1\|_2^2 + \|v_0^\lambda\|_2^2$ and $w_\lambda \in S(c_2)$. Also

$$A(w_\lambda) = A(\widetilde{u}_1) + A(v_0^\lambda) \text{ and } C(w_\lambda) = C(\widetilde{u}_1) + C(v_0^\lambda). \tag{3.5.11}$$

We claim that, for any $\lambda \in (0,1)$, there holds that

$$\Big| B(w_\lambda) - B(\widetilde{u}_1) - B(v_0^\lambda) \Big| \le \lambda \, \|\widetilde{u}_1\|_2^2 \cdot \|v_0^\lambda\|_2^2. \tag{3.5.12}$$

Indeed, from (3.5.10),

$$\Big(\widetilde{u}_1 + v_0^\lambda\Big)^2(x) = \widetilde{u}_1^2(x) + \Big(v_0^\lambda\Big)^2(x),\ \Big(\widetilde{u}_1 + v_0^\lambda\Big)^2(y) = \widetilde{u}_1^2(y) + \Big(v_0^\lambda\Big)^2(y).$$

Thus

$$\begin{aligned}
B(w_\lambda) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left(\widetilde{u}_1 + v_0^\lambda\right)^2(x)\cdot\left(\widetilde{u}_1 + v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\widetilde{u}_1^2(y)}{|x-y|}dxdy + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\left(v_0^\lambda\right)^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\
&= B(\widetilde{u}_1) + B(v_0^\lambda) + 2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy
\end{aligned}$$

with

$$\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy &= \int_{supp\ \widetilde{u}_1}\int_{supp\ v_0^\lambda} \frac{\widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)}{|x-y|}dxdy \\
&\leq \frac{\delta\lambda}{2(2-\lambda)}\int_{supp\ \widetilde{u}_1}\int_{supp\ v_0^\lambda} \widetilde{u}_1^2(x)\cdot\left(v_0^\lambda\right)^2(y)dxdy \\
&\leq \frac{\delta\lambda}{2(2-\lambda)}\|\widetilde{u}_1\|_2^2\cdot\left\|v_0^\lambda\right\|_2^2 \\
&\leq \frac{\lambda}{2}\|\widetilde{u}_1\|_2^2\cdot\left\|v_0^\lambda\right\|_2^2
\end{aligned}$$

and then (3.5.12) holds. Now from (3.5.11), (3.5.12) and using (3.5.9) we see that

$$A(w_\lambda) \to A(\widetilde{u}_1),\ B(w_\lambda) \to B(\widetilde{u}_1) \text{ and } C(w_\lambda) \to C(\widetilde{u}_1), \text{ as } \lambda \to 0. \tag{3.5.13}$$

Thus from Lemma 3.5.2 we have that, fixing $\lambda > 0$ small enough,

$$\max_{t>0} F(w_\lambda^t) \leq \max_{t>0} F(\widetilde{u}_1^t) + \frac{\varepsilon}{4}. \tag{3.5.14}$$

Now, using Lemma 3.5.1, (3.5.14), (3.5.8), (3.5.5) and (3.5.4) we have that

$$\begin{aligned}
\gamma(c_2) \leq \max_{t>0} F(w_\lambda^t) &\leq \max_{t>0} F(\widetilde{u}_1^t) + \frac{\varepsilon}{4} \\
&\leq \max_{t>0} F(u_1^t) + \frac{\varepsilon}{2} \\
&= F(u_1) + \frac{\varepsilon}{2} \leq \gamma(c_1) + \varepsilon,
\end{aligned}$$

and this ends the proof. □

Lemma 3.5.4. *When $p \in (\frac{10}{3}, 6)$, $c \to \gamma(c)$ is continuous at each $c > 0$.*

Proof. Since, by Lemma 3.5.3, $c \to \gamma(c)$ is non increasing proving that it is continuous at $c > 0$ is equivalent to show that for any sequence $c_n \to c^+$

$$\gamma(c) \leq \lim_{c_n \to c^+} \gamma(c_n). \tag{3.5.15}$$

Let $\varepsilon > 0$ be arbitrary but fixed. By Lemma 3.2.6 we know that there exists $u_n \in V(c_n)$ such that

$$F(u_n) \leq \gamma(c_n) + \frac{\varepsilon}{2}. \tag{3.5.16}$$

We define $\widetilde{u}_n = \sqrt{\frac{c}{c_n}} \cdot u_n := \rho_n \cdot u_n$. Then $\widetilde{u}_n \in S(c)$ and $\rho_n \to 1^-$. In addition

$$\begin{aligned}\gamma(c) &\leq \max_{t>0} F(\widetilde{u}_n^t)\\ &= \max_{t>0}\{\frac{t^2}{2}\rho_n^2 A(u_n) + \frac{t}{4}\rho_n^4 B(u_n) - \frac{t^{\frac{3p-6}{2}}}{p}\rho_n^p C(u_n)\}. \qquad (3.5.17)\end{aligned}$$

Since $u_n \in V(c_n)$ and $c_n \to c^+$, using the identity

$$F(u_n) - \frac{2}{3(p-2)}Q(u_n) = \frac{3p-10}{6(p-2)}A(u_n) + \frac{3p-8}{12(p-2)}B(u_n), \qquad (3.5.18)$$

it is not difficult to check that $A(u_n), B(u_n)$ and $C(u_n)$ are bounded both from above and from zero. Thus without restriction we can get that

$$A(u_n) \to A > 0,\ B(u_n) \to B \geq 0 \quad \text{and} \quad C(u_n) \to C > 0.$$

Indeed, $A \geq 0, B \geq 0, C \geq 0$ are trivial and it is also easy to verify by contradiction that $A \neq 0, C \neq 0$ from (3.4.6), (3.5.18) and the fact

$$Q(u_n) = A(u_n) + \frac{1}{4}B(u_n) - \frac{3p-6}{2p}C(u_n) = 0.$$

Now recording that $\rho_n \to 1^-$, using Lemma 3.5.2 twice, we get from (3.5.17), for any $n \in \mathbb{N}$ sufficiently large

$$\begin{aligned}\max_{t>0} F(\widetilde{u}_n^t) &\leq \max_{t>0}\{(\frac{A}{2})t^2 + (\frac{B}{4})t - (\frac{C}{p})t^{\frac{3}{2}(p-2)}\} + \frac{\varepsilon}{4}\\ &\leq \max_{t>0}\{(\frac{A(u_n)}{2})t^2 + (\frac{B(u_n)}{4})t - (\frac{C(u_n)}{p})t^{\frac{3}{2}(p-2)}\} + \frac{\varepsilon}{2}\\ &= \max_{t>0} F(u_n^t) + \frac{\varepsilon}{2} = F(u_n) + \frac{\varepsilon}{2}. \qquad (3.5.19)\end{aligned}$$

Now from (3.5.16) and (3.5.19) it follows that $\gamma(c) \leq \gamma(c_n) + \varepsilon$ for $n \in \mathbb{N}$ large enough and since $\varepsilon > 0$ is arbitrary (3.5.15) holds. □

Lemma 3.5.5. *Let $p \in (\frac{10}{3}, 6)$ and $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves*

$$-\Delta v - \lambda v + (|x|^{-1} * |v|^2)v - |v|^{p-2}v = 0 \ \textit{in}\ \mathbb{R}^3,$$

with $F(u_c) = \inf_{u \in V(c)} F(u) = \gamma(c)$. Then $\lambda_c \leq 0$ and moreover if $\lambda_c < 0$ the function $c \to \gamma(c)$ is strictly decreasing in a neighborhood of c.

Proof. To prove the lemma it suffices to show that if $\lambda_c < 0$ ($\lambda_c > 0$) the function $c \to \gamma(c)$ is strictly decreasing (increasing) in a neighborhood of c. Indeed, in view of Lemma 3.5.3 the case $\lambda_c > 0$ is then impossible.

The strict monotonicity of the function $c \to \gamma(c)$ when $\lambda_c \neq 0$ is obtained as a consequence of the Implicit Function Theorem.

Let us consider the following rescaled functions $u_{t,\theta}(x) = \theta^{\frac{3}{2}} t^{\frac{1}{2}} u_c(\theta x) \in S(tc)$ with $\theta \in (0, \infty)$ and $t \in (0, \infty)$. We define the following quantities

$$\alpha(t, \theta) = F(u_{t,\theta}), \qquad (3.5.20)$$

$$\beta(t, \theta) = Q(u_{t,\theta}). \qquad (3.5.21)$$

Simple calculus shows that

$$\frac{\partial \alpha(t,\theta)}{\partial t}_{|(1,1)} = \frac{1}{2}\Big(A(u_c) + B(u_c) - C(u_c)\Big) = \frac{1}{2}\lambda_c c \tag{3.5.22}$$

$$\frac{\partial \alpha(t,\theta)}{\partial \theta}_{|(1,1)} = 0, \qquad \frac{\partial^2 \alpha(t,\theta)}{\partial^2 \theta}_{|(1,1)} < 0. \tag{3.5.23}$$

Following the classical Lagrange Theorem we get, for any $\delta_t \in \mathbb{R}$, $\delta_\theta \in \mathbb{R}$,

$$\alpha(1+\delta_t, 1+\delta_\theta) = \alpha(1,1) + \delta_t \frac{\partial \alpha(t,\theta)}{\partial t}_{|(\bar{t},\bar{\theta})} + \delta_\theta \frac{\partial \alpha(t,\theta)}{\partial \theta}_{|(\bar{t},\bar{\theta})} \tag{3.5.24}$$

where $|1-\bar{t}| \leq |\delta_t|$ and $|1-\bar{\theta}| \leq |\delta_\theta|$, and by continuity, for sufficiently small $\delta_t > 0$ and sufficiently small $|\delta_\theta|$,

$$\alpha(1+\delta_t, 1+\delta_\theta) < \alpha(1,1) \quad \text{ if } \quad \lambda_c < 0 \tag{3.5.25}$$

$$\alpha(1-\delta_t, 1+\delta_\theta) < \alpha(1,1) \quad \text{ if } \quad \lambda_c > 0. \tag{3.5.26}$$

To conclude the proof it is enough to show that $\beta(t,u) = 0$ in a neighborhood of $(1,1)$ is the graph of a function $g : [1-\varepsilon, 1+\varepsilon] \to \mathbb{R}$ with $\varepsilon > 0$, such that $\beta(t, g(t)) = 0$ for $t \in [1-\varepsilon, 1+\varepsilon]$. Indeed in this case we have when $\lambda_c < 0$ by (3.5.25)

$$\gamma((1+\varepsilon)c) = \inf_{u \in V((1+\varepsilon)c)} F(u) \leq F(u_{1+\varepsilon, g(1+\varepsilon)}) < F(u_c) = \gamma(c)$$

and when $\lambda_c > 0$ we have by (3.5.26)

$$\gamma((1-\varepsilon)c) = \inf_{u \in V((1-\varepsilon)c)} F(u) \leq F(u_{1-\varepsilon, g(1-\varepsilon)}) < F(u_c) = \gamma(c).$$

To show the graph property by the Implicit Function Theorem it is sufficient to show that

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} \neq 0. \tag{3.5.27}$$

By simple calculus we get

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = 2A(u_c) + \frac{B(u_c)}{4} - \frac{1}{p}\Big(\frac{3}{2}(p-2)\Big)^2 C(u_c).$$

Using the fact that $Q(u_c) = 0$ we then obtain

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = (5 - \frac{3}{2}p)A(u_c) + (1 - \frac{3}{8}p)B(u_c).$$

Then, since $p > \frac{10}{3}$ we see that to have

$$\frac{\partial \beta(t,\theta)}{\partial \theta}_{|(1,1)} = 0$$

necessarily $A(u_c) = 0$ and $B(u_c) = 0$. Thus the derivative is never zero. □

Lemma 3.5.6. *Let $p \in (\frac{10}{3}, 6)$, then we have $\gamma(c) \to \infty$ as $c \to 0$.*

Proof. By Theorem 3.1.2 we know that for any $c > 0$ sufficiently small there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}^-$ solution of (E_λ) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. In addition by Lemma 3.4.3, $Q(u_c) = 0$. Thus $u_c \in H^1(\mathbb{R}^3)$ fulfills

$$0 = Q(u_c) = A(u_c) + \frac{1}{4}B(u_c) - \frac{3(p-2)}{2p}C(u_c) \tag{3.5.28}$$

$$\gamma(c) = F(u_c) = \frac{1}{2}A(u_c) + \frac{1}{4}B(u_c) - \frac{1}{p}C(u_c). \tag{3.5.29}$$

We deduce from (3.5.28) that $A(u_c) \leq -\frac{3(p-2)}{2p}C(u_c)$ and thus it follows from Gagliardo-Nirenberg's inequality, for some constant $C > 0$ that

$$\|\nabla u_c\|_2^2 \leq \frac{3(p-2)}{2p}\|u_c\|_p^p \leq C \cdot \|\nabla u_c\|_2^{\frac{3(p-2)}{2}} \cdot \|u_c\|_2^{\frac{6-p}{2}},$$

i.e

$$1 \leq C \cdot \|\nabla u_c\|_2^{\frac{3p-10}{2}} \cdot c^{\frac{6-p}{4}}. \tag{3.5.30}$$

Since $p \in (\frac{10}{3}, 6)$, we obtain that

$$\|\nabla u_c\|_2^2 \to \infty, \quad \text{as} \quad c \to 0. \tag{3.5.31}$$

Now from (3.5.28) and (3.5.29) we deduce that

$$\gamma(c) = F(u_c) = \frac{3p-10}{6(p-2)}A(u_c) + \frac{3p-8}{12(p-2)}B(u_c). \tag{3.5.32}$$

and thus from (6.2.18) we get immediately that $\gamma(c) \to \infty$ as $c \to 0$. □

3.6 Proof of Theorem 3.1.8 and Lemma 3.1.9

In this section we prove Theorem 3.1.8. Let us first show

Lemma 3.6.1. *Let $p \in (\frac{10}{3}, 6)$, for each $u_c \in \mathcal{M}_c$ there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solves (E_λ).*

Proof. From Lagrange multiplier theory, to prove the lemma, it suffices to show that any $u_c \in \mathcal{M}_c$ is a critical point of F constrained on $S(c)$.

Let $u_c \in \mathcal{M}_c$ and assume, by contradiction, that $\|F'|_{S(c)}(u_c)\|_{H^{-1}(\mathbb{R}^3)} \neq 0$. Then, by the continuity of F', there exist $\delta > 0, \mu > 0$ such that

$$v \in B_{u_c}(3\delta) \implies \|F'|_{S(c)}(v)\|_{H^{-1}(\mathbb{R}^3)} \geq \mu,$$

where $B_{u_c}(\delta) := \{v \in S(c) \ : \ \|v - u_c\| \leq \delta\}$.

Let $\varepsilon := \min\{\gamma(c)/4, \mu\delta/8\}$. We claim that it is possible to construct a deformation on $S(c)$ such that

(i) $\eta(1, v) = v$ if $v \notin F^{-1}([\gamma(c) - 2\varepsilon, \gamma(c) + 2\varepsilon])$,

(ii) $\eta(1, F^{\gamma(c)+\varepsilon} \cap B_{u_c}(\delta)) \subset F^{\gamma(c)-\varepsilon}$,

(iii) $F(\eta(1, v)) \leq F(v)$, $\forall\ v \in S(c)$.

Here, $F^d := \{u \in S(c) : F(u) \leq d\}$. For this we use the pseudo gradient flow on $S(c)$ defined in (3.3.5) but where now $g : S(c) \to [0, \delta]$ satisfies

$$g(v) := \begin{cases} \delta & \text{if } v \in B_{u_c}(2\delta) \cap F^{-1}([\gamma(c) - \varepsilon, \gamma(c) + \varepsilon]) \\ 0 & \text{if } v \notin F^{-1}([\gamma(c) - 2\varepsilon, \gamma(c) + 2\varepsilon]). \end{cases}$$

With this definition clearly (i) and (iii) hold. To prove (ii) first observe that if $v \in F^{\gamma(c)+\varepsilon} \cap B_{u_c}(\delta)$, then $\eta(t, v) \in B_{u_c}(2\delta)$ for all $t \in [0, 1]$. Indeed

$$\begin{aligned} \|\eta(t, v) - v\| &= \left\| \int_0^t -g(\eta(s, v)) \frac{Y(\eta(s, v))}{\|Y(\eta(s, v))\|} ds \right\| \\ &\leq \int_0^t \|g(\eta(s, v))\| \, ds \leq t\delta \leq \delta. \end{aligned}$$

In particular for $s \in [0, 1]$, $g(\eta(s, v)) = \delta$ as long as $F(\eta(s, v)) \geq \gamma(c) - \varepsilon$. Thus if we assume that there exists a $v \in F^{\gamma(c)+\varepsilon} \cap B_{u_c}(\delta)$ such that $F(\eta(1, v)) > \gamma(c) - \varepsilon$ we have

$$\begin{aligned} F(\eta(1, v)) &= F(v) + \int_0^1 \frac{d}{dt} F(\eta(t, v)) dt \\ &= F(v) + \int_0^1 \langle dF(\eta(t, v)), -g(\eta(t, v)) \frac{Y(\eta(t, v))}{\|Y(\eta(t, v))\|} \rangle dt \\ &\leq F(v) - \frac{\mu\delta}{4} \leq \gamma(c) + \varepsilon - \frac{\mu\delta}{4} \leq \gamma(c) - \varepsilon, \end{aligned}$$

i.e. $\eta(1, v) \in F^{\gamma(c)-\varepsilon}$. This contradiction proves that (ii) also hold.

Now let $g \in \Gamma_c$ be the path constructed in the proof of Lemma 3.2.6 by choosing $v = u_c \in V(c)$. We claim that

$$\max_{t \in [0,1]} F(\eta(1, g(t))) < \gamma(c). \tag{3.6.1}$$

By (i) and Remark 3.2.5 we have $\eta(1, g(t)) \in \Gamma_c$. Thus if (3.6.1) holds, it contradicts the definition of $\gamma(c)$. To prove (3.6.1), we distinguish three cases:
a) If $g(t) \in S(c) \setminus B_{u_c}(\delta)$, then using (iii) and Lemma 3.2.3 (6),

$$F(\eta(1, g(t))) \leq F(g(t)) < F(u_c) = \gamma(c).$$

b) If $g(t) \in F^{\gamma(c)-\varepsilon}$, then by (iii)

$$F(\eta(1, g(t))) \leq F(g(t)) \leq \gamma(c) - \varepsilon.$$

c) If $g(t) \in F^{-1}([\gamma(c) - \varepsilon, \gamma(c) + \varepsilon]) \cap B_{u_c}(\delta)$, then by (ii)

$$F(\eta(1, g(t))) \leq \gamma(c) - \varepsilon.$$

Note that since $F(g(t)) \leq \gamma(c)$, for all $t \in [0, 1]$ one of the three cases above must occurs. This proves that (3.6.1) hold and the proof of the lemma is completed. □

Proof of Theorem 3.1.8. We know from Lemma 3.6.1 that to each $u_c \in \mathcal{M}_c$ is associated a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ is solution of (E_λ). Now using Lemmas 3.5.5 we deduce that necessarily $\lambda_c \leq 0$. □

Proof of Lemma 3.1.9. Let $u_c \in H^1(\mathbb{R}^3, \mathbb{C})$ with $u_c \in V(c)$. Since $\|\nabla |u_c|\|_2 \leq \|\nabla u_c\|_2$ we have that $F(|u_c|) \leq F(u_c)$ and $Q(|u_c|) \leq Q(u_c) = 0$. In addition, by Lemma 3.2.3, there exists $t_0 \in (0,1]$ such that $Q(|u_c|^{t_0}) = 0$. We claim that

$$F(|u_c|^{t_0}) \leq t_0 \cdot F(u_c). \tag{3.6.2}$$

Indeed, due (3.2.2) and since $Q(|u_c|^{t_0}) = Q(u_c) = 0$, we have

$$\begin{aligned} F(|u_c|^{t_0}) &= t_0^2 \cdot \frac{3p-10}{6(p-2)} A(|u_c|) + t_0 \cdot \frac{3p-8}{12(p-2)} B(|u_c|) \\ &= t_0 \cdot \Big(t_0 \cdot \frac{3p-10}{6(p-2)} A(|u_c|) + \frac{3p-8}{12(p-2)} B(u_c) \Big) \\ &\leq t_0 \cdot \Big(\frac{3p-10}{6(p-2)} A(u_c) + \frac{3p-8}{12(p-2)} B(u_c) \Big) \\ &= t_0 \cdot F(u_c). \end{aligned}$$

Thus if $u_c \in H^1(\mathbb{R}^3, \mathbb{C})$ is a minimizer of F on $V(c)$ we have

$$F(u_c) = \inf_{u \in V(c)} F(u) \leq F(|u_c|^{t_0}) \leq t_0 \cdot F(u_c),$$

which implies $t_0 = 1$ since $t_0 \in (0,1]$. Then $Q(|u_c|) = 0$ and we conclude that

$$\|\nabla |u_c|\|_2 = \|\nabla u_c\|_2 \quad \text{and} \quad F(|u_c|) = F(u_c). \tag{3.6.3}$$

Thus point (i) follows. Now since $|u_c|$ is a minimizer of F on $V(c)$ we know by Theorem 3.1.8 that it satisfies (E_λ) for some $\lambda_c \leq 0$. By elliptic regularity theory and the maximum principle it follows that $|u_c| \in C^1(\mathbb{R}^3, \mathbb{R})$ and $|u_c| > 0$. At this point, using that $\|\nabla |u_c|\|_2 = \|\nabla u_c\|_2$ the rest of the proof of point (ii) is exactly the same as in the proof of [56, Theorem 4.1]. □

3.7 Proof of Theorems 3.1.4 and 3.1.6

In [63] the authors consider the functional F as a free functional defined in the real space

$$E := \Big\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy < \infty \Big\}$$

equipped with the norm

$$||u||_E := \Big(\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx + \Big(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x) u^2(y)}{|x-y|} dx dy \Big)^{\frac{1}{2}} \Big)^{\frac{1}{2}}.$$

Clearly $H^1(\mathbb{R}^3, \mathbb{R}) \subset E$. They show, see Theorem 1.1 and [63, Proposition 3.4], that F has in E a least energy solution whose energy is given by the mountain pass level

$$m := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} F(\gamma(t)) > 0 \tag{3.7.1}$$

where

$$\Gamma := \Big\{ \gamma \in C([0,1], E), \ \gamma(0) = 0, F(\gamma(1)) < 0 \Big\}.$$

Lemma 3.7.1. *For any $c > 0$ we have $\gamma(c) \geq m$ where $m > 0$ is given in (3.7.1).*

Proof. We fix an arbitrary $c > 0$. From Lemma 3.1.9 we know that the infimum of F on $V(c)$ is reached by real functions. As a consequence in the definition of $\gamma(c)$, see in particular (3.5.1), we can restrict ourself to paths in $H^1(\mathbb{R}^N, \mathbb{R})$ instead of $H^1(\mathbb{R}^N, \mathbb{C})$. To prove the lemma it suffices to show that for any $g \in \Gamma_c$ there exists a $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} F(g(t)) \geq \max_{t\in[0,1]} F(\gamma(t)). \tag{3.7.2}$$

Let $v \in S(c)$ be arbitrary but fixed. Letting $v^\theta(x) = \theta^{\frac{3}{2}} v(\theta x)$ we have $v^\theta \in S(c)$ for any $\theta > 0$. Also taking $\theta > 0$ sufficiently small, $v^\theta \in A_{K_c}$. Now for $g \in \Gamma_c$ arbitrary but fixed, let $\gamma_\theta(t) \in C([\frac{1}{4}, \frac{1}{2}], A_{K_c})$ satisfies $\gamma_\theta(\frac{1}{4}) = v^\theta, \gamma_\theta(\frac{1}{2}) = g(0)$, and consider $\gamma(t)$ given by

$$\gamma(t) := \begin{cases} 4tv^\theta, & 0 \leq t \leq 1/4, \\ \gamma_\theta(t), & 1/4 \leq t \leq 1/2, \\ g(2t-1), & 1/2 \leq t \leq 1. \end{cases}$$

Since $S(c) \subset H^1(\mathbb{R}^3) \subset E$ by construction $\gamma \in \Gamma$. Now direct calculations show that, taking $\theta > 0$ small enough, $F(4tv^\theta) \leq F(v^\theta)$ for any $t \in [0, \frac{1}{4}]$. Thus

$$\max_{t\in[0,1]} F(\gamma(t)) = \max_{t\in[\frac{1}{4},1]} F(\gamma(t)).$$

Recalling that $\gamma_\theta(t) \in A_{K_c}$ for any $t \in [\frac{1}{4}, \frac{1}{2}]$, we conclude from Theorem 3.2.1 that

$$\max_{t\in[0,1]} F(\gamma(t)) = \max_{t\in[\frac{1}{2},1]} F(\gamma(t)) = \max_{t\in[0,1]} F(g(t)),$$

and (3.7.2) holds. This proves the lemma. □

Lemma 3.7.2. *There exists $\gamma(\infty) > 0$ such that $\gamma(c) \to \gamma(\infty)$ as $c \to \infty$.*

Proof. The existence of a limit follows directly from the fact that $c \to \gamma(c)$ is non-increasing. Now because of Lemma 3.7.1 the limit is strictly positive. □

Proof of Theorem 3.1.6. As we already mentioned this proof is largely due to L. Dupaigne. It also uses arguments from [37] and [51]. We divide the proof into two steps.

Step 1: Regularity and vanishing: let (u, λ) with $u \in E$ and $\lambda \leq 0$ solves (E_λ), then $u \in L^\infty(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$ and $u(x) \to 0$, as $|x| \to \infty$.

We set $\phi_u(x) := \frac{1}{4\pi|x|} * u^2$. Clearly since $u \in E$ then $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. We denote $H = -\Delta + (1-\lambda)$. Since $\lambda \leq 0$, H^{-1} exists in $L^\eta(\mathbb{R}^3)$ for all $\eta \in (1, \infty)$. The operators H and $-\Delta$ being closed in $L^\eta(\mathbb{R}^3)$ with domain $D(H) \subset D(-\Delta)$, it follows from the Closed Graph Theorem that there exists a constant $\widetilde{C} > 0$ such that

$$\|\Delta u\|_\eta \leq \widetilde{C}\|Hu\|_\eta, \tag{3.7.3}$$

for any $u \in D(H)$. Now we write (E_λ) as

$$u = H^{-1}u - H^{-1}(\phi_u u) + H^{-1}(|u|^{p-2}u) \tag{3.7.4}$$

and we claim that

$$H^{-1}u \in L^3 \cap L^\infty(\mathbb{R}^3) \text{ and } H^{-1}(\phi_u u) \in L^2 \cap L^\infty(\mathbb{R}^3). \tag{3.7.5}$$

Indeed, $u \in L^q(\mathbb{R}^3)$ for all $q \in [3,6]$, see [100], and from (3.7.3) and the Sobolev embedding theorem, we obtain

$$H^{-1}u \in W^{2,q}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),\ \forall\, q \in [3,6]. \tag{3.7.6}$$

Now since $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, by Hölder inequality, $\phi_u u \in L^t(\mathbb{R}^3)$ holds for any $t \in [2,3]$ and we have

$$H^{-1}(\phi_u u) \in W^{2,t}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),\ \forall\, t \in [2,3]. \tag{3.7.7}$$

At this point the claim is proved. Next we denote

$$v := u + H^{-1}(\phi_u u) - H^{-1}u. \tag{3.7.8}$$

By interpolation, and using (3.7.5), we see that $v \in L^q(\mathbb{R}^3)$ for all $q \in [3,6]$. Now since $u \in L^q(\mathbb{R}^3)$, for all $q \in [3,6]$, (3.7.4) implies that

$$Hv = |u|^{p-2}u \in L^{\frac{q}{p-1}}(\mathbb{R}^3). \tag{3.7.9}$$

By (3.7.3) and Sobolev's embedding theorem, we conclude from (3.7.9) that

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \geq \frac{q}{p-1} \text{ such that } \frac{1}{r} \geq \frac{p-1}{q} - \frac{2}{3}. \tag{3.7.10}$$

Next we follow the arguments of T. Cazenave [37] to increase the index r.

For $j \geq 0$, we define r_j as:

$$\frac{1}{r_j} = -\delta(p-1)^j + \frac{2}{3(p-2)}, \quad \text{with } \delta = \frac{2}{3(p-2)} - \frac{1}{p}.$$

Since $p \in [3,6)$, then $\delta > 0$ and $\frac{1}{r_j}$ is decreasing with $\frac{1}{r_j} \to -\infty$ as $j \to \infty$. Thus there exists some $k > 0$ such that

$$\frac{1}{r_i} > 0 \ \text{ for } \ 0 \leq i \leq k; \ \frac{1}{r_{k+1}} \leq 0.$$

Now we claim that $v \in L^{r_k}(\mathbb{R}^3)$. Indeed, $r_0 = p$ as $j = 0$ and it is trivial that $v \in L^{r_0}(\mathbb{R}^3)$. If we assume that $v \in L^{r_i}(\mathbb{R}^3)$ for $0 \leq i < k$, then by (3.7.8) and (3.7.5), we have $u \in L^{r_i}(\mathbb{R}^3)$. Thus following (3.7.10) we obtain

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \geq \frac{r_i}{p-1} \text{ such that } \frac{1}{r} \geq \frac{p-1}{r_i} - \frac{2}{3} = \frac{1}{r_{i+1}}.$$

In particular, $v \in L^{r_{i+1}}(\mathbb{R}^3)$ and we conclude this claim by induction. Now since $v \in L^{r_k}(\mathbb{R}^3)$ it follows from (3.7.8) and (3.7.5) that $u \in L^{r_k}(\mathbb{R}^3)$ and we get that

$$v \in L^r(\mathbb{R}^3), \quad \text{for all } r \geq \frac{r_k}{p-1} \text{ such that } \frac{1}{r} \geq \frac{p-1}{r_k} - \frac{2}{3} = \frac{1}{r_{k+1}}.$$

Since $1/r_{k+1} < 0$ we obtain that $v \in \cap_{3 \leq \alpha \leq \infty} L^\alpha(\mathbb{R}^3)$ and thus also $u \in \cap_{3 \leq \alpha \leq +\infty} L^\alpha(\mathbb{R}^3)$.

At this point we have shown that

$$Hu = u - \phi_u u + |u|^{p-2}u$$

with for all $\alpha \in [3, \infty]$,

$$u \in L^{\alpha} \cap L^{\infty}(\mathbb{R}^3), \quad \phi_u u \in L^{\frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \quad \text{and} \quad |u|^{p-2}u \in L^{\frac{\alpha}{p-1}} \cap L^{\infty}(\mathbb{R}^3).$$

Since $\frac{6\alpha}{6+\alpha} \in [3, 6]$ for $\alpha \in [6, \infty]$, by interpolation and (3.7.3) we obtain that

$$u \in W^{2,\frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \text{ for any } \alpha \in [6, +\infty]. \tag{3.7.11}$$

Thus by Sobolev's embedding, $u \in L^{\infty}(\mathbb{R}^3) \cap C^1(\mathbb{R}^3)$. Also there exists a sequence $\{u_n\} \subset C_c^1(\mathbb{R}^3)$ such that $u_n \to u$ in $W^{2,\frac{6\alpha}{6+\alpha}}(\mathbb{R}^3)$. When $\alpha > 6$, $W^{2,\frac{6\alpha}{6+\alpha}}(\mathbb{R}^3) \hookrightarrow L^{\infty}(\mathbb{R}^3)$. Thus $u_n \to u$ uniformly in $\mathbb{R}^3$ and we conclude that $u(x) \to 0$ as $|x| \to \infty$.

Step 2: Exponential decay estimate.

First we show that $\phi_u \in C^{0,\gamma}(\mathbb{R}^3)$, $\forall \gamma \in (0, 1)$ and that there exists a constant $C_0 > 0$ such that

$$\phi_u \geq \frac{C_0}{|x|}, \quad \text{for all } |x| \geq 1. \tag{3.7.12}$$

Since $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ solves the equation $-\Delta\Phi = 4\pi|u|^2$ and $u \in L^6(\mathbb{R}^3)$ by elliptic regularity $\phi_u \in W^{2,3}_{loc}(\mathbb{R}^3)$. Thus by Sobolev's embedding, $\phi_u \in C^{0,\gamma}(\mathbb{R}^3)$, $\forall \gamma \in (0, 1)$. In particular

$$C_0 = \min_{\partial B_1} \phi_u(x) > 0$$

where $B_R := \{x \in \mathbb{R}^3 : |x| \leq R\}$. Indeed, if $\phi_u(x_0) = 0$ at some point $x_0 \in \mathbb{R}^3$ with $|x_0| = 1$, then $u(x) = 0$ a.e. in $\mathbb{R}^3$.

Now for an arbitrary $R_0 > 0$, let $w_1 := \phi_u - \frac{C_0}{|x|}$. Then

$$\begin{cases} -\Delta w_1 = 4\pi u^2 \geq 0, & \text{in } B_{R_0} \setminus B_1, \\ w_1 \geq 0, & \text{on } \partial B_1, \\ w_1 \geq -\frac{C_0}{R}, & \text{on } \partial B_{R_0}, \end{cases}$$

and the maximum principle yields that

$$w_1 \geq -\frac{C_0}{R_0}, \quad \text{in } B_{R_0} \setminus B_1.$$

Letting $R_0 \to \infty$, it follows that $w_1 \geq 0$ in $\mathbb{R}^3 \setminus B_1$ and thus (3.7.12) holds.

By Kato's inequality, we know that $\Delta u^+ \geq \chi[u \geq 0]\Delta u$, see [25]. Thus

$$-\Delta u^+ - \lambda u^+ + \phi_u \cdot u^+ \leq (u^+)^{p-1} \text{ in } \mathbb{R}^3. \tag{3.7.13}$$

Let us show that there exist constants $\widetilde{C} > 0$ and $R_1 > 0$ such that

$$u^+(x) \leq \widetilde{C}\phi_u(x) \text{ for } |x| > R_1. \tag{3.7.14}$$

To prove this, we consider $w_2 := u^+ - \phi_u - \frac{d}{|x|}$, for a constant $d > 0$. Then (3.7.13) and $\lambda \leq 0$ imply that

$$-\Delta w_2 \leq (u^+)^{p-1} - 4\pi u^2, \quad \text{in } |x| \geq 1.$$

Since $\lim_{|x|\to\infty} u(x) \to 0$ and $p > 3$, then $(u^+)^{p-1} - 4\pi u^2 \leq 0$ holds in $|x| \geq R_1$ for some $R_1 > 0$ large enough. Thus for any $R \geq R_1$ and taking $d > 0$ large enough we have

$$\begin{cases} -\Delta w_2 \leq 0, & \text{in } B_R \setminus B_{R_1}; \\ w_2 \leq 0, & \text{on } \partial B_{R_1}; \\ w_2 \leq \max\limits_{\partial B_R} u^+ - \frac{d}{R}, & \text{on } \partial B_R. \end{cases}$$

Then by the maximum principle, we have $w_2 \leq \max_{\partial B_R} u^+ - \frac{d}{R}$ in $B_R \setminus B_{R_1}$. Letting $R \to \infty$ we conclude that $w_2 \leq 0$ in $\mathbb{R}^3 \setminus B_{R_1}$. This, together with (3.7.12), implies (3.7.14).

From (3.7.13) we have for any $\sigma > 0$ and since $\lambda \leq 0$,

$$\begin{aligned} -\Delta u^+ + \frac{\sigma}{|x|}u^+ &\leq \frac{\sigma}{|x|}u^+ - \phi_u u^+ + \lambda u^+ + (u^+)^{p-1} \\ &\leq \left(\frac{\sigma}{|x|} - \phi_u + (u^+)^{p-2}\right) u^+. \end{aligned} \tag{3.7.15}$$

Using (3.7.12) and (3.7.14), for $|x| \geq R_1 > 1$, by choosing $0 < \sigma < C_0$, we have

$$\begin{aligned} \frac{\sigma}{|x|} - \phi_u + (u^+)^{p-2} &\leq \frac{\sigma}{C_0} \cdot \phi_u - \phi_u + (u^+)^{p-2} \\ &\leq -(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} u^+ + (u^+)^{p-2} \\ &= \left(-(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} + (u^+)^{p-3}\right) \cdot u^+, \end{aligned}$$

where $(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} > 0$. Since $p \geq 3$ and $u(x) \to 0$ as $|x| \to \infty$, for $R_1 > 1$ sufficiently large, we obtain that $-(1 - \frac{\sigma}{C_0})\widetilde{C}^{-1} + (u^+)^{p-3} \leq 0$ in $|x| \geq R_1$. Thus it follows from (3.7.15) that

$$-\Delta u^+ + \frac{\sigma}{|x|}u^+ \leq 0, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1}. \tag{3.7.16}$$

If we denote $\bar{C}_1 := \max_{\partial B_{R_1}} u^+$, applying the maximum principle, we thus obtain

$$u^+ \leq \bar{C}_1 \cdot \bar{w}, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1} \tag{3.7.17}$$

where $\bar{w}$ is the radial solution of

$$\begin{cases} -\Delta \bar{w} + \frac{\sigma}{|x|} = 0, & \text{if } |x| > R_1; \\ \bar{w}(x) = 1, & \text{if } |x| = R_1; \\ \bar{w}(x) \to 0, & \text{if } |x| \to \infty. \end{cases}$$

Now $\bar{w}$ satisfies (see [3, Section 4]),

$$\bar{w}(x) \leq C \cdot |x|^{-3/4} e^{-2C'\sqrt{|x|}}, \quad \forall\, |x| > R', \tag{3.7.18}$$

for some $C > 0, C' > 0$ and $R' > 0$.

Finally we observe that if u is a solution of (E_λ), then $-u$ is also a solution. Thus since $u^- = (-u)^+$, following the same arguments, we obtain that there exists a constant $\bar{C}_2 > 0$ such that

$$u^- \leq \bar{C}_2 \cdot \bar{w}, \quad \text{in } \mathbb{R}^3 \setminus B_{R_1}. \tag{3.7.19}$$

Hence $|u| = u^+ + u^- \leq (\bar{C}_1 + \bar{C}_2)\bar{w}$, in $\mathbb{R}^3 \setminus B_{R_1}$ for $R_1 > 0$ sufficiently large. At this point we see from (3.7.18) that $u \in E$ satisfies the exponential decay (3.1.6). In particular $u \in L^2(\mathbb{R}^3)$ and then also $u \in H^1(\mathbb{R}^3)$. □

Lemma 3.7.3. *There exists $c_\infty > 0$ such that for all $c \geq c_\infty$ the function $c \to \gamma(c)$ is constant. Also if for a $c \geq c_\infty$ there exists a couple $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ solution of (E_λ) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$, then necessarily $\lambda_c = 0$.*

Proof. From Lemma 3.1.9 and [63] we know that there exist grounds states of the free functional F which are real. Also Theorem 3.1.6 implies that any ground state belongs to $H^1(\mathbb{R}^3)$. Let $u_0 \in H^1(\mathbb{R}^3)$ be one of these ground states and set $c_0 := ||u_0||_2^2$. Then, by Lemma 3.4.3, $u_0 \in V(c_0)$ and using Lemma 3.7.1 we get

$$F(u_0) \geq \gamma(c_0) \geq m = F(u_0).$$

Thus necessarily $\gamma(c_0) = m$. Now since $c \to \gamma(c)$ is non increasing, still by Lemma 3.7.1, we deduce that $\gamma(c) = \gamma(c_0)$ for all $c \geq c_0$. Now let $(u_c, \lambda_c) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ be a solution of (E_λ) with $||u_c||_2^2 = c$ and $F(u_c) = \gamma(c)$. Thus by Lemma 3.5.5, $\lambda_c \leq 0$. But we note that $\lambda_c < 0$ will not happen since by Lemma 3.5.5 it would imply that $c \to \gamma(c)$ is strictly decreasing around $c > 0$ in contradiction with the fact that $\gamma(c)$ is constant. Then necessarily $\lambda_c = 0$. □

Remark 3.7.4. We see, from Theorem 3.1.8 and Lemma 3.7.3, that if $\gamma(c)$ is reached, say by a $u_c \in H^1(\mathbb{R}^3)$ with $c > 0$ large enough, then u_c is a ground state of F defined on E. It is unlikely that ground states exist for an infinity of value of $c > 0$. So we conjecture that there exists a $c_{lim} > 0$ such that for $c \geq c_{lim}$ there are no critical points for F constrained to $S(c)$ at the ground state level $\gamma(c)$.

Proof of Theorem 3.1.4. Obviously, points (i), (ii), (iv), (v) of Theorem 3.1.4 follow directly from Lemmas 3.5.3, 3.5.4, 3.5.6, 3.7.2, 3.7.3 and Lemmas 3.4.3, 3.5.5 derive point (iii). □

3.8 Global existence and strong instability

We introduce the following result about the local well-posedness of the Cauchy problem of (3.1.1) (see T. Cazenave [37, Theorem 4.4.6 and Proposition 6.5.1], or H. Kikuchi [72, Chapter 3]).

Proposition 3.8.1. *Let $p \in (2, 6)$, for any $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$, there exist a $T = T(\|u_0\|) > 0$ and a unique solution $u(t) \in C([0, T), H^1(\mathbb{R}^3, \mathbb{C}))$ of the equation (3.1.1) with initial datum $u(0) = u_0$ satisfying*

$$F(u(t)) = F(u_0),\ \|u(t)\|_2 = \|u_0\|_2\,,\ \forall\ t \in [0, T).$$

In addition, if $u_0 \in H^1(\mathbb{R}^3, \mathbb{C})$ satisfies $|x|u_0 \in L^2(\mathbb{R}^3, \mathbb{C})$, then the virial identity

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\|xu(t)\|_2^2 = 8Q(u),$$

holds for any $t \in [0, T)$.

Proof of Theorem 3.1.11. Let $u(x, t)$ be the solution of (3.1.1) with $u(x, 0) = u_0$ and $T_{max} \in (0, \infty]$ its maximal time of existence. Then classically we have either

$$T_{max} = +\infty$$

or

$$T_{max} < +\infty \quad \text{and} \lim_{t \to T_{max}} ||\nabla u(x, t)||_2^2 = \infty. \tag{3.8.1}$$

Since

$$F(u(x,t)) - \frac{2}{3(p-2)}Q(u(x,t)) = \frac{3p-10}{6(p-2)}A(u(x,t)) + \frac{3p-8}{12(p-2)}B(u(x,t))$$

and $F(u(x,t)) = F(u_0)$ for all $t < T_{max}$, if (3.8.1) happens then, we get

$$\lim_{t \to T_{max}} Q(u(x,t)) = -\infty.$$

By continuity it exists $t_0 \in (0, T_{max})$ such that $Q(u(x,t_0)) = 0$ with $F(u(x,t_0)) = F(u_0) < \gamma(c)$. This contradicts the definition $\gamma(c) = \inf_{u \in V(c)} F(u)$. $\square$

Remark 3.8.2. For $p \in (\frac{10}{3}, 6)$ and any $c > 0$ the set $\mathcal{O}$ is not empty. Indeed for an arbitrary but fixed $u \in S(c)$, let $u^t(x) = t^{\frac{3}{2}} u(tx)$. Then $u^t \in S(c)$ for all $t > 0$ and

$$\begin{aligned} Q(u^t) &= t^2 A(u) + \frac{t}{4} B(u) - \frac{3(p-2)}{2p} t^{\frac{3(p-2)}{2}} C(u), \\ F(u^t) &= \frac{t^2}{2} A(u) + \frac{t}{4} B(u) - \frac{t^{\frac{3(p-2)}{2}}}{p} C(u). \end{aligned}$$

We observe that $F(u^t) \to 0$ as $t \to 0$. Also, since $\frac{3(p-2)}{2} > 1$, we have $Q(u^t) > 0$ when $t > 0$ is sufficiently small. This proves that $\mathcal{O}$ is not empty.

Proof of the Theorem 3.1.12. For any $c > 0$, let $u_c \in \mathcal{M}_c$ and define the set

$$\Theta := \left\{ v \in H^1(\mathbb{R}^3) \setminus \{0\} : \ F(v) < F(u_c), \ \|v\|_2^2 = \|u_c\|_2^2, \ Q(v) < 0 \right\}.$$

The set Θ contains elements arbitrary close to u_c in $H^1(\mathbb{R}^3)$. Indeed, letting $v_0(x) = u_c^\lambda = \lambda^{\frac{3}{2}} u_c(\lambda x)$, with $\lambda < 1$, we see from Lemma 3.2.3 that $v_0 \in \Theta$ and that $v_0 \to u_c$ in $H^1(\mathbb{R}^3)$ as $\lambda \to 1$.

Let $v(t)$ be the maximal solution of (3.1.1) with initial datum $v(0) = v_0$ and $T \in (0, \infty]$ the maximal time of existence. Let us show that $v(t) \in \Theta$ for all $t \in [0, T)$. From the conservation laws, we have that

$$\|v(t)\|_2^2 = \|v_0\|_2^2 = \|u_c\|_2^2, \ \text{ and } \ F(v(t)) = F(v_0) < F(u_c).$$

Thus it is enough to verify $Q(v(t)) < 0$. But $Q(v(t)) \neq 0$ for any $t \in (0, T)$. Otherwise, by the definition of $\gamma(c)$, we would get for a $t_0 \in (0, T)$ that $F(v(t_0)) \geq F(u_c)$ in contradiction with $F(v(t)) < F(u_c)$. Now by continuity of Q we get that $Q(v(t)) < 0$ and thus that $v(t) \in \Theta$ for all $t \in [0, T)$. Now we claim that there exists $\delta > 0$, such that

$$Q(v(t)) \leq -\delta, \ \forall t \in [0, T). \tag{3.8.2}$$

Let $t \in [0, T)$ be arbitrary but fixed and set $v = v(t)$. Since $Q(v) < 0$ we know by Lemma 3.2.3 that $\lambda^\star(v) < 1$ and that $\lambda \mapsto F(v^\lambda)$ is concave on $[\lambda^\star, 1)$. Hence

$$\begin{aligned} F(v^{\lambda^\star}) - F(v) &\leq (\lambda^\star - 1) \frac{\partial}{\partial \lambda} F(v^\lambda) \mid_{\lambda=1} \\ &= (\lambda^\star - 1) Q(v). \end{aligned}$$

Thus, since $Q(v(t)) < 0$, we have

$$F(v) - F(v^{\lambda^\star}) \geq (1 - \lambda^\star) Q(v) \geq Q(v).$$

It follows from $F(v) = F(v_0)$ and $v^{\lambda^\star} \in V(c)$ that

$$Q(v) \leq F(v) - F(v^{\lambda^\star}) \leq F(v_0) - F(u_c).$$

Then letting $\delta = F(u_0) - F(v_0) > 0$ the claim is established. To end the proof of the theorem we next use Proposition 3.8.1. Since $v_0(x) = u_c^\lambda$ we have that

$$\int_{\mathbb{R}^3} |x|^2 |v_0|^2 dx = \int_{\mathbb{R}^3} |x|^2 |u_c^\lambda|^2 dx = \lambda^2 \int_{\mathbb{R}^3} |y|^2 |u_c(y)|^2 dy.$$

Thus, from Lemma 3.6.1 and Theorem 3.1.6, we obtain that

$$\int_{\mathbb{R}^3} |x|^2 |v_0|^2 dx < \infty. \tag{3.8.3}$$

Applying Proposition 3.8.1 it follows that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \|xv(t)\|_2^2 = 8Q(v).$$

Now by (3.8.2) we deduce that $v(t)$ must blow-up in finite time, namely that (3.8.1) hold. Recording that v_0 has been taken arbitrarily close to u_c, this ends the proof of the theorem. □

Proof of Theorem 3.1.15. For $p \in (\frac{10}{3}, 6)$, let u_0 be a ground state of equation (3.1.4). From Theorem 3.1.6 we know that $u_0 \in H^1(\mathbb{R}^3)$, thus we can set

$$c_0 := \|u_0\|_2^2.$$

From Lemma 3.4.3, we have $Q(u_0) = 0$. Thus $u_0 \in V(c_0)$ and it follows from (3.1.3) and Lemma 3.7.1 that

$$F(u_0) \geq \gamma(c_0) \geq m = F(u_0).$$

Hence $F(u_0) = \inf_{u \in V(c_0)} F(u)$, which means that u_0 minimizes F on $V(c_0)$. Thus applying Theorem 3.1.12, we are done with the proof. □

3.9 Comparison with the nonlinear Schrödinger case

In [64] the existence of critical points of

$$\tilde{F}(u) := \frac{1}{2} \|\nabla u\|_2^2 - \frac{1}{p}\|u\|_p^p, \quad u \in H^1(\mathbb{R}^N). \tag{3.9.1}$$

constrained to $S(c)$ was considered under the condition:

$$(C) : \frac{2N+4}{N} < p < \frac{2N}{N-2}, \text{ if } N \geq 3 \text{ and } \frac{2N+4}{N} < p \text{ if } N = 1, 2.$$

In our notation it is proved in [64] that $\tilde{F}$ has a mountain pass geometry on $S(c)$ in the sense that

$$\tilde{\gamma}(c) = \inf_{g \in \tilde{\Gamma}_c} \max_{t \in [0,1]} \tilde{F}(g(t)) > \max\{\tilde{F}(g(0)), \tilde{F}(g(1))\} > 0,$$

where

$$\tilde{\Gamma}_c = \Big\{g \in C([0,1], S(c)),\ g(0) \in A_{K_c}, \tilde{F}(g(1)) < 0\Big\},$$

and $A_{K_c} = \{u \in S(c) : \ \|\nabla u\|_2^2 \leq K_c\}$. Also we have

Lemma 3.9.1. *([64, Theorem 2]) For $N \geq 1$ and any $c > 0$, under the condition (C), the functional $\tilde{F}$ admits a critical point u_c at the level $\tilde{\gamma}(c)$ with $\|u_c\|_2^2 = c$, and there exists $\lambda_c < 0$ such that (u_c, λ_c) solves weakly the following Euler-Lagrange equation associated with $\tilde{F}$:*

$$-\Delta u - \lambda u = |u|^{p-2}u. \tag{3.9.2}$$

Lemma 3.9.2. *([64, Corollary 3.1 and Theorem 3.2]) For $N \geq 1$,*

$$\begin{cases} \|\nabla u_c\|_2^2 \to \infty, \\ \lambda_c \to -\infty, \end{cases} \quad as \quad c \to 0,$$

and

$$\begin{cases} \|\nabla u_c\|_2^2 \to 0, \\ \lambda_c \to 0, \end{cases} \quad as \quad c \to \infty.$$

Using the above two results we now prove

Lemma 3.9.3. *For $N \geq 1$, under the condition (C), the function $c \mapsto \tilde{\gamma}(c)$ is strictly decreasing. In addition, we have*

$$\begin{cases} \tilde{\gamma}(c) \to +\infty, & as\ c \to 0, \\ \tilde{\gamma}(c) \to 0, & as\ c \to \infty. \end{cases} \tag{3.9.3}$$

Proof. Arguing as in the proof of Lemma 3.5.1 we can deduce that

$$\tilde{\gamma}(c) = \inf_{u \in S(c)} \max_{t>0} \tilde{F}(u^t) = \inf_{u \in \tilde{V}(c)} \tilde{F}(u). \tag{3.9.4}$$

Here $\tilde{V}(c) := \{u \in H^1(\mathbb{R}^N) : \tilde{Q}(u) = 0\}$ with

$$\tilde{Q}(u) = \|\nabla u\|_2^2 - \frac{N(p-2)}{2p}\|u\|_p^p,$$

and $u^t(x) := t^{\frac{N}{2}}u(tx)$ for $t > 0$. To show that $c \to \tilde{\gamma}(c)$ is strictly decreasing we just need to prove that: for any $c_1 < c_2$, there holds $\tilde{\gamma}(c_2) < \tilde{\gamma}(c_1)$. By (3.9.4) we have

$$\tilde{\gamma}(c_1) = \inf_{u \in S(c_1)} \max_{t>0} \tilde{F}(u^t) \quad \text{and} \quad \tilde{\gamma}(c_2) = \inf_{u \in S(c_2)} \max_{t>0} \tilde{F}(u^t)$$

where

$$\tilde{F}(u^t) = \frac{t^2}{2}\|\nabla u\|_2^2 - \frac{t^{\frac{N}{2}(p-2)}}{p}\|u\|_p^p.$$

After a simple calculation, we get

$$\max_{t>0} \tilde{F}(u^t) = \tilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|\nabla u\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \tag{3.9.5}$$

with

$$\tilde{c}(p) = \left(\frac{4}{N(p-2)}\right)^{\frac{4}{N(p-2)-4}} \cdot \frac{N(p-2)-4}{N(p-2)} > 0.$$

By Lemma 3.9.1, we know that $\gamma(c_1)$ is attained, namely that there exists $u_1 \in S(c_1)$, such that $\tilde{\gamma}(c_1) = \tilde{F}(u_1) = \max_{t>0} \tilde{F}(u_1^t)$. Then using the scaling $u_\theta(x) = \theta^{1-\frac{N}{2}}u_1(\frac{x}{\theta})$, we have

$$\|u_\theta\|_2^2 = \theta^2\|u_1\|_2^2, \quad \|\nabla u_\theta\|_2^2 = \|\nabla u_1\|_2^2 \quad \text{and} \quad \|u_\theta\|_p^p = \theta^{(1-\frac{N}{2})p+N}\|u_1\|_p^p.$$

Thus we can choose $\theta > 1$ such that $u_\theta \in S(c_2)$. Under the condition (C), we have $(1-\frac{N}{2})p + N > 0$ for $N \geq 1$ and thus $\|u_\theta\|_p^p > \|u_1\|_p^p$. Now we have

$$\begin{aligned}
\max_{t>0} \tilde{F}(u_\theta^t) &= \widetilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u_\theta\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|u_\theta\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \\
&< \widetilde{c}(p) \cdot \left(\frac{1}{2}\|\nabla u_1\|_2^2\right)^{\frac{N(p-2)}{N(p-2)-4}} \cdot \left(\frac{1}{p}\|u_1\|_p^p\right)^{-\frac{4}{N(p-2)-4}} \\
&= \max_{t>0} \tilde{F}(u_1^t),
\end{aligned}$$

which implies that

$$\tilde{\gamma}(c_1) = \max_{t>0} \tilde{F}(u_1^t) > \max_{t>0} \tilde{F}(u_\theta^t) \geq \tilde{\gamma}(c_2). \tag{3.9.6}$$

Finally, from [64, Lemma 2.7] we know that, for any $c > 0$, $\tilde{Q}(u_c) = 0$. Thus we can write

$$\tilde{\gamma}(c) = \frac{N(p-2)-4}{2N(p-2)} \|\nabla u_c\|_2^2$$

and then (3.9.3) directly follows from Lemma 3.9.2. □

Finally in analogy to Theorems 3.1.8 and 3.1.12 we have

Remark 3.9.4. Let

$$\tilde{\mathcal{M}}_c := \{u_c \in \tilde{V}(c) \ : \ \tilde{F}(u_c) = \inf_{u \in \tilde{V}(c)} \tilde{F}(u)\}. \tag{3.9.7}$$

Then for any $u_c \in \tilde{\mathcal{M}}_c$ there exists a $\lambda_c < 0$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves (3.9.2) and the standing wave solution $e^{-i\lambda_c t}u_c$ of (3.1.8) is strongly unstable.

The proof of these statements is actually simpler than the ones for (3.1.1) and thus we just indicate the main lines. We proceed as in Lemma 3.6.1 to show that for any $u_c \in \tilde{\mathcal{M}}_c$ there exists a $\lambda_c \in \mathbb{R}$ such that $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ solves (3.9.2). Indeed a version of Lemma 3.2.3 (and thus of Lemma 3.2.6) holds when $\tilde{F}$ replaces F and this is precisely [37, Lemma 8.2.5]. Now if for a $\lambda \in \mathbb{R}$, $u \in S(c)$ solves

$$-\Delta u - |u|^{p-2}u = \lambda u, \tag{3.9.8}$$

on one hand, multiplying (3.9.8) by $u \in S(c)$ and integrating we obtain

$$||\nabla u||_2^2 - ||u||_p^p = \lambda c. \tag{3.9.9}$$

On the other hand, since solutions of (3.9.8) satisfy $\tilde{Q}(u) = 0$, we have

$$||\nabla u||_2^2 - \frac{N(p-2)}{2p}||u||_p^p = 0. \tag{3.9.10}$$

Thus, under the condition (C), since $N(p-2)/2p < 1$, we deduce that necessarily $\lambda < 0$. To conclude the proof we just have to show that the standing wave $e^{-i\lambda_c t}u_c$ is strongly unstable. This can be done by following the same lines as in the proof of Theorem 3.1.12. Here the fact that $\lambda_c < 0$ insures the exponential decay at infinity of $u_c \in S(c)$ which permits to use the virial identity in the blow-up argument (see also [18]).

Chapter 4

Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations

4.1 Introduction

In this chapter, we establish a multiplicity result for the stationary Schrödinger-Poisson-Slater equations

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0 \text{ in } \mathbb{R}^3. \tag{E_λ}$$

Our result is the following.

Theorem 4.1.1. *Assume that $p \in (\frac{10}{3}, 6)$. There exists a $c_0 > 0$ such that for any $c \in (0, c_0)$, the equation (E_λ) admits an unbounded sequence of distinct pairs of solutions $(\pm u_n, \lambda_n)$ with $\|u_n\|_2^2 = c$ and $\lambda_n < 0$ for each $n \in \mathbb{N}$.*

Clearly the sequence of solutions $(\pm u_n, \lambda_n) \in H^1(\mathbb{R}^3) \times \mathbb{R}$ will be obtained as critical points and associated Lagrange multipliers of the functional

$$F(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|} dxdy - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p dx, \tag{4.1.1}$$

on the constraint

$$S(c) := \Big\{u \in H^1(\mathbb{R}^3) : \ \|u\|_2^2 = c, c > 0\Big\}. \tag{4.1.2}$$

In view that F is unbounded from below on $S(c)$ when $p \in (\frac{10}{3}, 6)$, the genus of the sublevel set $F^\alpha := \{u \in S(c) : F(u) \leq \alpha\}$ is always infinite. Thus to obtain the existence of infinitely many solutions, classical arguments based on the Kranoselski genus, see [107], do not apply.

Since we are not concerned here, as it was the case in Chapters 2 and 3, by the search of least energy solutions, we can work in the subspace $H_r^1(\mathbb{R}^3)$ of radially symmetric functions. It is classical that a critical point of F restricted to $H_r^1(\mathbb{R}^3) \cap S(c)$ is a critical point of F

restricted to $H^1(\mathbb{R}^3)\cap S(c)$. The advantage of working in $H^1_r(\mathbb{R}^3)$ is that the embedding of $H^1_r(\mathbb{R}^3)$ into $L^q(\mathbb{R}^3)$ is compact for $q\in(2,6)$. However, as it can easily be checked, despite this property, F restricted to $S(c)$ does not satisfy the Palais-Smale condition.

To overcome these difficulties we rely on a recent work of T. Bartsch and S. De Valeriola [12]. In [12] the authors consider the problem of finding infinitely many critical points for

$$\mathcal{E}(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx, \tag{4.1.3}$$

on the constraint

$$S_r(c) := \Big\{u\in H^1_r(\mathbb{R}^3):\ \|u\|_2^2 = c, c>0\Big\}, \tag{4.1.4}$$

when $p\in(\frac{10}{3},6)$. Actually in [12] more general nonlinearities can be handled and in any dimension $N\geq 2$.

In the problem treated in [12] the difficulties presented above already exist. To overcome these difficulties the authors present a new type of linking geometry for the functional $\mathcal{E}$ on $S_r(c)$. This geometry is, according to the authors of [12], motivated by the fountain theorem (see [11]). In [12] to set up a min-max scheme and identify a sequence $\{l_n\}\subset\mathbb{R}$, $l_n\to\infty$ of suspected critical levels, the cohomological index for spaces with an action on the group $G=\{-1,1\}$ is used. Indeed observe that the functional $\mathcal{E}$ is even, this is also the case of F. This index which was introduced in [31] permits to establish the key intersection property, see [12, Lemma 2.3] or our Lemma 4.2.3. The fact that the suspected critical levels l_n do correspond to critical levels is then obtained using ideas from [64]. The key point is the construction, for each fixed $n\in\mathbb{N}$, of a bounded Palais-Smale sequence associated with l_n. In that aim one introduces an auxiliary functional which permits to incorporate into the variational procedure the information that any critical point of $\mathcal{E}$ on $S_r(c)$ must satisfy a version of Pohozaev identity. Having obtained the boundedness of a Palais-Smale sequence it remains to show that it converges. The information that the associated Lagrange multiplier is strictly negative is here crucially used.

In our proof of Theorem 4.1.1 we follow closely the strategy of [12]. The restriction that $c\in(0,c_0)$ originates in the need to show that the suspected associated Lagrange multipliers are strictly negative. This property is used to show that the weak limit of our Palais-Smale sequences does belong to $S_r(c)$. A similar limitation on $c>0$ was already necessary in the last chapter for the existence of just one critical point. More generally the present chapter makes a strong use of results derived in Chapter 3.

Up to our knowledge, Theorem 4.1.1 is the first result in the literature on the existence of infinitely many L^2-normalized solutions for equation (E_λ). Previous results had already been obtained when $\lambda\in\mathbb{R}$ is a fixed parameter. We refer to [4, 8, 42, 104] and their references in that direction.

4.2 Proofs of the main results

We first establish some preliminary results. Let $\{V_n\}\subset H^1_r(\mathbb{R}^3)$ be a strictly increasing sequence of finite-dimensional linear subspaces in $H^1_r(\mathbb{R}^3)$, such that $\bigcup_n V_n$ is dense in $H^1_r(\mathbb{R}^3)$. We denote by $V_n^\perp$ the orthogonal space of V_n in $H^1_r(\mathbb{R}^3)$. Then

Lemma 4.2.1. *[12, Lemma 2.1] Assume that $p\in(2,6)$. Then there holds*

$$\mu_n := \inf_{u\in V_{n-1}^\perp}\frac{\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx}{(\int_{\mathbb{R}^3}|u|^p dx)^{2/p}} = \inf_{u\in V_{n-1}^\perp}\frac{\|u\|^2}{\|u\|_p^2}\to\infty, \quad \textit{as } n\to\infty.$$

Now for $c > 0$ fixed and for each $n \in \mathbb{N}$, we define

$$\rho_n := L^{-\frac{2}{p-2}} \cdot \mu_n^{\frac{2}{p-2}}, \quad \text{with } L = \max_{x>0} \frac{(x^2+c)^{p/2}}{x^p + c^{p/2}},$$

and

$$B_n := \{u \in V_{n-1}^{\perp} \cap S_r(c) : \ \|\nabla u\|_2^2 = \rho_n\}. \tag{4.2.1}$$

We also define

$$b_n := \inf_{u \in B_n} F(u). \tag{4.2.2}$$

Then we have

Lemma 4.2.2. *For any $p \in (2,6)$, $b_n \to +\infty$ as $n \to \infty$. In particular we can assume without restriction that $b_n \geq 1$ for all $n \in \mathbb{N}$.*

Proof. For any $u \in B_n$, we have that

$$\begin{aligned} F(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^2|u(y)|^2}{|x-y|}dxdy - \frac{1}{p}\int_{\mathbb{R}^3}|u|^p dx \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p\mu_n}\Big(\|\nabla u\|_2^2 + c\Big)^{\frac{p}{2}} \\ &\geq \frac{1}{2}\|\nabla u\|_2^2 - \frac{L}{p\mu_n}\Big(\|\nabla u\|_2^p + c^{\frac{p}{2}}\Big) \\ &\geq (\frac{1}{2} - \frac{1}{p})\rho_n - \frac{L}{p\mu_n}c^{\frac{p}{2}}. \end{aligned}$$

From this estimate and Lemma 4.2.1, it follows since $p > 2$, that $b_n \to +\infty$ as $n \to \infty$. Now, considering the sequence $\{V_n\} \subset H_r^1(\mathbb{R}^3)$ only from a $n_0 \in \mathbb{N}$ such that $b_n \geq 1$ for any $n \geq n_0$ it concludes the proof of the lemma. $\square$

Next we start to set up our min-max scheme. First we introduce the map

$$\begin{aligned} \kappa : H_r^1(\mathbb{R}^3) \times \mathbb{R} &\longrightarrow H_r^1(\mathbb{R}^3) \\ (u,\ \theta) &\longmapsto \kappa(u,\theta)(x) := e^{\frac{3}{2}\theta}u(e^{\theta}x). \end{aligned} \tag{4.2.3}$$

Observe that for any given $u \in S_r(c)$, we have $\kappa(u,\theta) \in S_r(c)$ for all $\theta \in \mathbb{R}$. Also from Lemma 3.2.2, we know that

$$\begin{cases} A(\kappa(u,\theta)) \to 0, \quad F(\kappa(u,\theta)) \to 0, \qquad \text{as } \theta \to -\infty, \\ A(\kappa(u,\theta)) \to +\infty, \ F(\kappa(u,\theta)) \to -\infty, \quad \text{as } \theta \to +\infty. \end{cases} \tag{4.2.4}$$

Thus, using the fact that V_n is finite dimensional, we deduce that, for each $n \in \mathbb{N}$, there exists a $\theta_n > 0$, such that

$$\bar{g}_n : \ [0,1] \times (S_r(c) \cap V_n) \to S_r(c), \ \ \bar{g}_n(t,u) = \kappa(u,(2t-1)\theta_n) \tag{4.2.5}$$

satisfies

$$\begin{cases} A(\bar{g}_n(0,u)) < \rho_n, \ A(\bar{g}_n(1,u)) > \rho_n, \\ F(\bar{g}_n(0,u)) < b_n, \ F(\bar{g}_n(1,u)) < b_n. \end{cases} \tag{4.2.6}$$

Now we define

$$\Gamma_n := \Big\{ g : [0,1] \times (S_r(c) \cap V_n) \to S_r(c) \mid g \text{ is continuous, odd in } u \tag{4.2.7}$$

$$\text{and such that } \forall u : \; g(0,u) = \bar{g}_n(0,u), \; g(1,u) = \bar{g}_n(1,u) \Big\}. \tag{4.2.8}$$

Clearly $\bar{g}_n \in \Gamma_n$. Now we give the key intersection result, due to [12].

Lemma 4.2.3. *For each $n \in \mathbb{N}$,*

$$\gamma_n(c) := \inf_{g \in \Gamma_n} \max_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} F(g(t,u)) \ge b_n. \tag{4.2.9}$$

Proof. The point to show that for each $g \in \Gamma_n$ there exists a pair $(t,u) \in [0,1] \times (S_r(c) \cap V_n)$, such that $g(t,u) \in B_n$ with B_n defined in (4.2.1). But this result can be proved exactly as the corresponding result for J in [12], see [12, Lemma 2.3]. □

Remark 4.2.4. Note that by Lemma 4.2.3 and (4.2.6) we have that for any $g \in \Gamma_n$

$$\gamma_n(c) \ge b_n > \max \Big\{ \max_{u \in S_r(c) \cap V_n} F(g(0,u)), \max_{u \in S_r(c) \cap V_n} F(g(1,u)) \Big\}.$$

Next, we shall prove that the sequence $\{\gamma_n(c)\}$ is indeed a sequence of critical values for F restricted to $S_r(c)$. In this aim, we first show that there exists a bounded Palais-Smale sequence at each level $\gamma_n(c)$. From now on, we fix an arbitrary $n \in \mathbb{N}$.

Lemma 4.2.5. *For any fixed $c > 0$, there exists a sequence $\{u_k\} \subset S_r(c)$ satisfying*

$$\begin{cases} F(u_k) \to \gamma_n(c), \\ F'|_{S_r(c)}(u_k) \to 0, \qquad \text{as } k \to \infty, \\ Q(u_k) \to 0, \end{cases} \tag{4.2.10}$$

where

$$Q(u) := A(u) + \frac{1}{4} B(u) - \frac{3(p-2)}{2p} C(u). \tag{4.2.11}$$

In particular $\{u_k\} \subset S_r(c)$ is bounded.

To find such a Palais-Smale sequence, we apply the approach developed by L. Jeanjean [64], already applied in [12]. First, we introduce the auxiliary functional

$$\widetilde{F} : S_r(c) \times \mathbb{R} \to \mathbb{R}, \qquad (u,\theta) \mapsto F(\kappa(u,\theta)),$$

where $\kappa(u,\theta)$ is given in (4.2.3), and the set

$$\widetilde{\Gamma}_n := \Big\{ \widetilde{g} : [0,1] \times (S_r(c) \cap V_n) \to S_r(c) \times \mathbb{R} \mid \widetilde{g} \text{ is continuous, odd in } u,$$

$$\text{and such that } \kappa \circ \widetilde{g} \in \Gamma_n \Big\}.$$

Clearly, for any $g \in \Gamma_n$, $\widetilde{g} := (g, 0) \in \widetilde{\Gamma}_n$.

Observe that defining

$$\widetilde{\gamma}_n(c) := \inf_{\widetilde{g} \in \widetilde{\Gamma}_n} \max_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} \widetilde{F}(\widetilde{g}(t,u)),$$

we have that $\widetilde{\gamma}_n(c) = \gamma_n(c)$. Indeed, by the definitions of $\widetilde{\gamma}_n(c)$ and $\gamma_n(c)$, this identity follows immediately from the fact that the maps

$$\varphi : \Gamma_n \longrightarrow \widetilde{\Gamma}_n,\ g \longmapsto \varphi(g) := (g, 0),$$

and

$$\psi : \widetilde{\Gamma}_n \longrightarrow \Gamma_n,\ \widetilde{g} \longmapsto \psi(\widetilde{g}) := \kappa \circ \widetilde{g},$$

satisfy

$$\widetilde{F}(\varphi(g)) = F(g) \ \text{ and } \ F(\kappa \circ \widetilde{g}) = \widetilde{F}(\widetilde{g}).$$

To prove Lemma 4.2.5 we also need the following result, which was established by Ekeland's variational principle in [64, Lemma 2.3]. We denote by E the set $H_r^1(\mathbb{R}^3) \times \mathbb{R}$ equipped with $\|\cdot\|_E^2 = \|\cdot\|^2 + |\cdot|_{\mathbb{R}}^2$, and by E^* its dual space.

Lemma 4.2.6. *Let $\varepsilon > 0$. Suppose that $\widetilde{g}_0 \in \widetilde{\Gamma}_n$ satisfies*

$$\max_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} \widetilde{F}(\widetilde{g}_0(t, u)) \le \widetilde{\gamma}_n(c) + \varepsilon.$$

Then there exists a pair of $(u_0, \theta_0) \in S_r(c) \times \mathbb{R}$ such that:

(1) $\widetilde{F}(u_0, \theta_0) \in [\widetilde{\gamma}_n(c) - \varepsilon, \widetilde{\gamma}_n(c) + \varepsilon]$*;*

(2) $\displaystyle\min_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} \|(u_0, \theta_0) - \widetilde{g}_k(t, u)\|_E \le \sqrt{\varepsilon}$*;*

(3) $\|\widetilde{F}'|_{S_r(c) \times \mathbb{R}}(u_0, \theta_0)\|_{E^*} \le 2\sqrt{\varepsilon}$*, i.e.*

$$|\langle \widetilde{F}'(u_0, \theta_0), z \rangle_{E^* \times E}| \le 2\sqrt{\varepsilon}\, \|z\|_E\,,$$

holds for all $z \in \widetilde{T}_{(u_0, \theta_0)} := \{(z_1, z_2) \in E, \langle u_0, z_1 \rangle_{L^2} = 0\}$.

Now we can give

Proof of Lemma 4.2.5. From the definition of $\gamma_n(c)$, we know that for each $k \in \mathbb{N}$, there exists a $g_k \in \Gamma_n$ such that

$$\max_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} F(g_k(t, u)) \le \gamma_n(c) + \frac{1}{k}.$$

Since $\widetilde{\gamma}_n(c) = \gamma_n(c)$, $\widetilde{g}_k = (g_k, 0) \in \widetilde{\Gamma}_n$ satisfies

$$\max_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} \widetilde{F}(\widetilde{g}_k(t, u)) \le \widetilde{\gamma}_n(c) + \frac{1}{k}.$$

Thus applying Lemma 4.2.6, we obtain a sequence $\{(u_k, \theta_k)\} \subset S_r(c) \times \mathbb{R}$ such that:

(i) $\widetilde{F}(u_k, \theta_k) \in [\gamma_n(c) - \frac{1}{k}, \gamma_n(c) + \frac{1}{k}]$;

(ii) $\displaystyle\min_{\substack{0 \le t \le 1 \\ u \in S_r(c) \cap V_n}} \|(u_k, \theta_k) - (g_k(t, u), 0)\|_E \le \frac{1}{\sqrt{k}}$;

(iii) $\|\widetilde{F}'|_{S_r(c)\times\mathbb{R}}(u_k,\theta_k)\|_{E^*} \leq \frac{2}{\sqrt{k}}$, i.e.

$$|\langle \widetilde{F}'(u_k,\theta_k), z\rangle_{E^*\times E}| \leq \frac{2}{\sqrt{k}}\|z\|_E\,,$$

holds for all $z \in \widetilde{T}_{(u_k,\theta_k)} := \{(z_1,z_2)\in E, \langle u_k, z_1\rangle_{L^2} = 0\}$.

For each $k \in \mathbb{N}$, let $v_k = \kappa(u_k,\theta_k)$. We shall prove that $v_k \in S_r(c)$ satisfies (4.2.10). Indeed, first, from (i) we have that $F(v_k) \underset{k}{\to} \gamma_n(c)$, since $F(v_k) = F(\kappa(u_k,\theta_k)) = \widetilde{F}(u_k,\theta_k)$. Secondly, note that

$$Q(v_k) = A(v_k) + \frac{1}{4}B(v_k) - \frac{3(p-2)}{2p}C(v_k) = \langle \widetilde{F}'(u_k,\theta_k),(0,1)\rangle_{E^*\times E},$$

and $(0,1)\in \widetilde{T}_{(u_k,\theta_k)}$. Thus (iii) yields $Q(v_k) \underset{k}{\to} 0$. Finally, to verify that $F'|_{S_r(c)}(v_k) \underset{k}{\to} 0$, it suffices to prove for $k\in\mathbb{N}$ sufficiently large, that

$$|\langle F'(v_k), w\rangle_{(H_r^1)^*\times H_r^1}| \leq \frac{4}{\sqrt{k}}\|w\|^2\,, \quad \text{for all} \quad w \in T_{v_k}, \tag{4.2.12}$$

where $T_{v_k} := \{w \in H_r^1(\mathbb{R}^3),\ \langle v_k, w\rangle_{L^2} = 0\}$. To this end, we note that, for $w \in T_{v_k}$, setting $\widetilde{w} = \kappa(w,-\theta_k)$, one has

$$\begin{aligned}
\langle F'(v_k), w\rangle_{(H_r^1)^*\times H_r^1} &= \int_{\mathbb{R}^3}\nabla v_k\nabla w dx + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v_k(x)|^2 v_k(y)w(y)}{|x-y|}dxdy - \int_{\mathbb{R}^3}|v_k|^{p-2}v_k w dx\\
&= e^{2\theta_k}\int_{\mathbb{R}^3}\nabla u_k\nabla\widetilde{w}dx + e^{\theta_k}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u_k(x)|^2 u_k(y)\widetilde{w}(y)}{|x-y|}dxdy - e^{\frac{3(p-2)}{2}\theta_k}\int_{\mathbb{R}^3}|u_k|^{p-2}u_k\widetilde{w}dx\\
&= \langle \widetilde{F}'(u_k,\theta_k),(\widetilde{w},0)\rangle_{E^*\times E} \qquad .
\end{aligned}$$

If $(\widetilde{w},0) \in \widetilde{T}_{(u_k,\theta_k)}$ and $\|(\widetilde{w},0)\|_E^2 \leq 2\|w\|^2$ when $k\in\mathbb{N}$ is sufficiently large, then (iii) implies (4.2.12). To verify these conditions, observes that $(\widetilde{w},0)\in\widetilde{T}_{(u_k,\theta_k))} \Leftrightarrow w \in T_{v_k}$. Also from (ii) it follows that

$$|\theta_k| = |\theta_k - 0| \leq \min_{\substack{0\leq t\leq 1\\ u\in S_r(c)\cap V_n}}\|(v_k,\theta_k)-(g_k(t,u),0)\|_E \leq \frac{1}{\sqrt{k}},$$

by which we deduce that

$$\|(\widetilde{w},0)\|_E^2 = \|\widetilde{w}\|^2 = \int_{\mathbb{R}^3}|w(x)|^2dx + e^{-2\theta_k}\int_{\mathbb{R}^3}|\nabla w(x)|^2dx \leq 2\|w\|^2,$$

holds for $k\in\mathbb{N}$ large enough. At this point, (4.2.12) has been verified. To end the proof of the lemma it remains to show that $\{v_k\}\subset S_r(c)$ is bounded. But since $p\in(\frac{10}{3},6)$ this follows immediately from the following relationship between $F(u)$ and $Q(u)$,

$$F(u) - \frac{2}{3(p-2)}Q(u) = \frac{3p-10}{6(p-2)}A(u) + \frac{3p-8}{12(p-2)}B(u). \tag{4.2.13}$$

□

Remark 4.2.7. Note that in Lemma 2.2.1, it is proved that any critical point $u_0\in S_r(c)$ of F on $S_r(c)$ must satisfy $Q(u_0)=0$. So far this information has been used in Lemma 4.2.5 to construct a bounded Palais-Smale sequence. As we shall see in our next result it is also useful to insure that our Palais-Smale sequences do not vanish.

Proposition 4.2.8. *Let $\{u_k\} \subset S_r(c)$ be the Palais-Smale sequence obtained in Lemma 4.2.5. Then there exist $\lambda_n \in \mathbb{R}$ and $u_n \in H_r^1(\mathbb{R}^3)$, such that, up to a subsequence,*

i) $u_k \rightharpoonup u_n \neq 0$, *in* $H_r^1(\mathbb{R}^3)$,

ii) $-\Delta u_k - \lambda_n u_k + (|x|^{-1} * |u_k|^2)u_k - |u_k|^{p-2}u_k \to 0$, *in* $H_r^{-1}(\mathbb{R}^3)$,

iii) $-\Delta u_n - \lambda_n u_n + (|x|^{-1} * |u_n|^2)u_n - |u_n|^{p-2}u_n = 0$, *in* $H_r^{-1}(\mathbb{R}^3)$.

Moreover, if $\lambda_n < 0$, then we have

$$u_k \to u_n, \quad \text{in } H_r^1(\mathbb{R}^3), \quad \text{as } k \to \infty.$$

In particular, $||u_n||_2^2 = c$, $F(u_n) = \gamma_n(c)$ and $F'(u_n) - \lambda_n u_n = 0$ in $H_r^{-1}(\mathbb{R}^3)$.

Proof. Since $\{u_k\} \subset S_r(c)$ is bounded, up to a subsequence, there exists a $u_n \in H_r^1(\mathbb{R}^3)$, such that

$$u_k \underset{k}{\rightharpoonup} u_n, \quad \text{in } H_r^1(\mathbb{R}^3),$$
$$u_k \underset{k}{\to} u_n, \quad \text{in } L^p(\mathbb{R}^3).$$

We have $u_n \neq 0$. Indeed suppose by contradiction that $u_n = 0$. Then by the strong convergence in $L^p(\mathbb{R}^3)$ it follows that $C(u_k) \to 0$. Taking into account that $Q(u_k) \to 0$ it then implies that $A(u_k) \to 0$ and $B(u_k) \to 0$. Thus $F(u_k) \to 0$ and this contradicts the fact that $\gamma_n(c) \geq b_n \geq 1$. Thus Point *i)* holds.

The proofs of Points *ii)* and *iii)* can be found in Proposition 3.4.1. Now using Points *ii)*, *iii)*, and the convergence $C(u_k) \underset{k}{\to} C(u_n)$, it follows that

$$A(u_k) - \lambda_n D(u_k) + B(u_k) \underset{k}{\to} A(u_n) - \lambda_n D(u_n) + B(u_n).$$

If $\lambda_n < 0$, then we conclude from the weak convergence of $u_k \underset{k}{\rightharpoonup} u_n$ in $H_r^1(\mathbb{R}^3)$, that

$$A(u_k) \underset{k}{\to} A(u_n),\ B(u_k) \underset{k}{\to} B(u_n),\ C(u_k) \underset{k}{\to} C(u_n).$$

Thus $u_k \underset{k}{\to} u_n$ in $H_r^1(\mathbb{R}^3)$, and in particular, $||u_n||_2^2 = c$, $F(u_n) = \gamma_n(c)$ and $F'(u_n) - \lambda_n u_n = 0$ in $H_r^{-1}(\mathbb{R}^3)$. □

At this point we can prove our main result.

Proof of Theorem 4.1.1. By Lemma 4.2.5 and Proposition 4.2.8, to prove Theorem 4.1.1, it is enough to verify that if $(u_n, \lambda_n) \in S_r(c) \times \mathbb{R}$ solves

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^3,$$

then necessarily $\lambda_n < 0$ provided $c > 0$ is sufficiently small. However, this point has been proved in Lemma 3.4.3 of Chapter 3. Thus the proof of the theorem is completed. □

Chapter 5

Sharp non-existence results of normalized solutions for a quasi-linear Schrödinger equation

5.1 Introduction

The aim of this chapter is to clarify and extend some results contained in [41] where a constrained minimization problem associated with a quasi-linear equation is considered. We recall that in [41], the authors study the existence of solutions for the following stationary quasi-linear Schrödinger equation

$$-\Delta u - \lambda u - u\Delta(u^2) - |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^N, \tag{P_λ}$$

where $\lambda \in \mathbb{R}$, $p \in (1, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (1, \infty)$ if $N = 1, 2$. Solutions of (P_λ) are found by considering the following minimization problem

$$\bar{m}(c) := \inf_{u \in \bar{S}(c)} J(u), \; c > 0, \tag{5.1.1}$$

where

$$J(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx, \tag{5.1.2}$$

and

$$\bar{S}(c) := \Big\{ u \in \mathcal{X} : \; \|u\|_2^2 = c \Big\} \text{ with } \mathcal{X} = \{u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx < \infty\}.$$

It is standard that if a $u_c \in \bar{S}(c)$ minimizes globally J on $\bar{S}(c)$, then there exists a $\lambda_c \in \mathbb{R}$ such that the couple (u_c, λ_c) solves (P_λ).

By means of (1.2.7), it is not difficult to conclude that for any $c > 0$, $\bar{m}(c) > -\infty$ if $p < 3 + \frac{4}{N}$, and $\bar{m}(c) = -\infty$ when $p > 3 + \frac{4}{N}$. Hence the minimization problem (5.1.1) is only studied as $p \in (1, 3 + \frac{N}{4}]$. In [41], it is proved that when $p \in (1, 1 + \frac{N}{4})$, for all $c > 0$, $\bar{m}(c)$ admits a minimizer. When $p \in (1 + \frac{N}{4}, 3 + \frac{N}{4}]$, it is claimed that there exists a $c(p, N) > 0$, such that minimizers of $\bar{m}(c)$ do not exist for $c < c(p, N)$, but do exist for $c > c(p, N)$. However, there are some gaps in the proofs of [41].

In this chapter, we focus on the range $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N}]$, and try to clarify and extend some results in [41]. In particular, we settle the question of existence for the threshold value $c(p,N)$. Before presenting our main results, we first give a detailed study of the property of the function $c \to \bar{m}(c)$. This study is of interest for itself, but also essential for the establishment of our sharp non-existence results. Let

$$c(p,N) := \inf\{c > 0 : \ \bar{m}(c) < 0\}.$$

Theorem 5.1.1. *(i) If $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$, we have*

a) $c(p,N) \in (0,\infty)$;

b) $\bar{m}(c) = 0$ if $c \in (0, c(p,N)]$;

c) $\bar{m}(c) < 0$ if $c \in (c(p,N),\infty)$ and is strictly decreasing about c, as $c \in (c(p,N),\infty)$.

(ii) If $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$, the mapping $c \mapsto \bar{m}(c)$ is continuous at each $c > 0$.

(iii) If $p = 3+\frac{4}{N}$, we denote

$$c_N := \inf\{c > 0 \ : \ \exists\ u \in \bar{S}(c) \text{ such that } J(u) \leq 0\}, \tag{5.1.3}$$

then $c_N \in (0,\infty)$ and

$$\begin{cases} \bar{m}(c) = 0, & \text{as } c \in (0,\ c_N); \\ \bar{m}(c) = -\infty, & \text{as } \ c \in (c_N, \infty). \end{cases} \tag{5.1.4}$$

Concerning the existence or non-existence of minimizers we have

Theorem 5.1.2. *(i) If $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$, then $\bar{m}(c)$ admits a minimizer if and only if $c \in [c(p,N),\infty)$.*

(ii) If $p = 3+\frac{4}{N}$, $\bar{m}(c)$ has no minimizer for all $c \in (0,\infty)$.

Remark 5.1.3. We note that in [41] it was proved that when $p \in (1, 1+\frac{4}{N})$, for all $c > 0$, $\bar{m}(c) < 0$ and $\bar{m}(c)$ admits a minimizer. When $p = 1+\frac{4}{N}$, we conjecture that the conclusion of Theorem 5.1.2 (i) also holds.

Remark 5.1.4. We point out that parts of Theorems 5.1.1 and 5.1.2 are already contained in [41, Theorem 1.12]. However, on one hand we provide here additional information. In particular we settle the question of existence for the threshold value $c(p,N)$ which requires a special treatment. On the other hand some statements of [41, Theorem 1.12] are wrong, in particular concerning the case $p = 3+\frac{4}{N}$. There are also some gaps in the proofs of [41]. In particular it is not proved completely that there are no minimizer when $c \in (0, c(p,N))$.

Remark 5.1.5. In [32], the minimization problem (5.1.1) is studied and the question of finding explicit bounds on $c(p,N)$ and c_N is addressed by a combination of analytical and numerical arguments in dimension $N = 3$. In particular, when $p = 3+\frac{4}{N}$ a $c_b > 0$ such that $\bar{m}(c) = 0$ if $0 < c \leq c_b$ and a $c^b > 0$ such that $\bar{m}(c) = -\infty$ if $c > c^b$ are explicitly given (see [32, Proposition 2.1, points (4) and (5)]). Their values are $c_b \approx 19.73$ and $c^b \approx 85.09$. Theorem 5.1.1 (iii) complements these results showing that the change from $\bar{m}(c) = 0$ to $\bar{m}(c) = -\infty$ occurs abruptly at the value c_N. We also point out that our results hold for any dimension $N \in \mathbb{N}$.

Finally, similarly to Theorem 2.1.6 we prove

Theorem 5.1.6. *Assume that $p \in [1+\frac{4}{N}, 3+\frac{4}{N}]$ holds, then there exists a $\hat{c} > 0$ such that for all $c \in (0,\hat{c})$, the functional J, restricted to $\bar{S}(c)$, has no critical points.*

5.2 Preliminary Results

First we observe

Lemma 5.2.1. *Assume that* $p \in [1+\frac{4}{N}, 3+\frac{4}{N})$. *If there exists a* $\bar{c} > 0$ *such that* $\bar{m}(\bar{c}) = 0$ *is achieved, then*

$$\bar{m}(c) < 0, \quad \textit{for all} \ \ c > \bar{c}. \tag{5.2.1}$$

Proof. Let $\bar{u} \in \bar{S}(\bar{c})$ be a minimizer of $\bar{m}(\bar{c})$. Setting $(\bar{u})_t(x) = \bar{u}(t^{-\frac{1}{N}}x)$ for $t > 1$, we have $\|(\bar{u})_t\|_2^2 = t\|\bar{u}\|_2^2 = t\bar{c}$, and

$$\begin{aligned} \bar{m}(t\bar{c}) \le J((\bar{u})_t) &= t^{1-\frac{2}{N}} \left(\int_{\mathbb{R}^N} \frac{1}{2}|\nabla \bar{u}|^2 + |\bar{u}|^2|\nabla \bar{u}|^2 dx \right) - \frac{t}{p+1} \int_{\mathbb{R}^N} |\bar{u}|^{p+1} dx \\ &= t \left[t^{-\frac{2}{N}} \int_{\mathbb{R}^N} \left(\frac{1}{2}|\nabla \bar{u}|^2 + |\bar{u}|^2|\nabla \bar{u}|^2 \right) dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |\bar{u}|^{p+1} dx \right] \\ &< tJ(\bar{u}) = t\bar{m}(\bar{c}). \end{aligned} \tag{5.2.2}$$

Thus (5.2.1) follows immediately from (5.2.2) since $\bar{m}(\bar{c}) = 0$. □

Let c_N be given by (5.1.3), then we have

Lemma 5.2.2. *Assume that* $p = 3 + \frac{4}{N}$. *Then* $c_N \in (0, \infty)$.

Proof. We know from (1.2.7) that when $p \in [1+\frac{4}{N}, 3+\frac{4}{N}]$ there exists a $C > 0$, depending only on p and N, such that

$$\|u\|_{p+1}^{p+1} \le C \cdot \|u\|_2^{2(1-\theta)} \cdot \left(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx \right)^{\frac{\theta N}{N-2}}, \ \forall \ \in \mathcal{X}, \tag{5.2.3}$$

where $\theta = \frac{(p-1)(N-2)}{2(N+2)}$. Letting $p = 3 + \frac{4}{N}$ in (5.2.3), we obtain that

$$\|u\|_{4+4/N}^{4+4/N} \le C \cdot \|u\|_2^{\frac{4}{N}} \cdot \left(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx \right), \quad \text{for all } u \in \mathcal{X}. \tag{5.2.4}$$

Thus, for any $u \in \bar{S}(c)$, there holds

$$\begin{aligned} J(u) &\ge \frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - C \cdot c^{\frac{2}{N}} \cdot \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx \\ &\ge \left(1 - C \cdot c^{\frac{2}{N}}\right) \cdot \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx \end{aligned}$$

and $J(u) > 0$ for all $u \in \bar{S}(c)$ if $c > 0$ is sufficiently small. This proves that $c_N > 0$.

Now take $u_1 \in \bar{S}(1)$ arbitrary and consider the scaling

$$u_t(x) := u_1(t^{-\frac{1}{N}x}), \quad \text{for all } t > 0. \tag{5.2.5}$$

We have $u_t \in \bar{S}(t)$ and

$$\begin{aligned} J(u_t) &= t^{1-\frac{2}{N}} \left(\frac{1}{2}\|\nabla u_1\|_2^2 + \int_{\mathbb{R}^N} |u_1|^2|\nabla u_1|^2 dx \right) - t \cdot \frac{N}{4(N+1)}\|u_1\|_{4+4/N}^{4+4/N} \\ &= t \left[t^{-\frac{2}{N}} \left(\frac{1}{2}\|\nabla u_1\|_2^2 + \int_{\mathbb{R}^N} |u_1|^2|\nabla u_1|^2 dx \right) - \frac{N}{4(N+1)}\|u_1\|_{4+4/N}^{4+4/N} \right]. \end{aligned} \tag{5.2.6}$$

This shows that $J(u_t) < 0$ for $t > 0$ large and proves that $c_N < \infty$. □

5.3 Special treatments for the limit case $c = c(p, N)$

Now we treat particularly the limit case $c = c(p, N)$.

Lemma 5.3.1. *Assume that $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$. Then $\bar{m}(c(p, N))$ admits a minimizer.*

Proof. Let $c_n := c(p, N) + 1/n$, for all $n \in \mathbb{N}$. Since $\bar{m}(c_n) < 0$ we know by [41, Lemma 4.4] that $\bar{m}(c_n)$ admits, for all $n \in \mathbb{N}$ a minimizer that is Schwarz symmetric. We claim that $\{u_n\}$ is bounded in $\mathcal{X}$, namely that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$ is bounded. Indeed using (5.2.3) we have since $J(u_n) \le 0$, for all $n \in \mathbb{N}^+$,

$$\begin{aligned} \frac{1}{2}\|\nabla u_n\|_2^2 + \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx &\le \frac{1}{p+1} \int_{\mathbb{R}^N} |u_n|^{p+1} dx \\ &\le \frac{C}{p+1} c_n^{1-\theta} \cdot \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \right)^{\frac{\theta N}{N-2}}. \quad (5.3.1) \end{aligned}$$

Since $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N})$ we have $\frac{\theta N}{N-2} < 1$ and thus (5.3.1) implies that both $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$ and $\{||\nabla u_n||_2^2\}$ are bounded.

Passing to a subsequence we can assume that $u_n \rightharpoonup u_0$ in $\mathcal{X}$. Now from [41, Lemma 4.3] we have that

$$Z(u_0) \le \liminf_{n \to \infty} Z(u_n) \quad \text{where} \quad Z(u) := \frac{1}{2} \|\nabla u\|_2^2 + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx.$$

Also the fact that $\{u_n\}$ is a sequence of Schwarz symmetric functions readily implies that $u_n \to u_0$ in $L^{p+1}(\mathbb{R}^N)$. Thus, since by Theorem 5.1.1 (ii), $\lim_{n\to\infty} J(u_n) = \lim_{n\to\infty} \bar{m}(c_n) = 0$ we obtain that $J(u_0) \le 0$. Also since $||u_0||_2^2 \le c(p, N)$ necessarily $J(u_0) = 0$.

In order to show that $||u_0||_2^2 = c(p, N)$ and thus that u_0 is a minimizer of $c(p, N)$ we first show that $u_0 \ne 0$. By contradiction let us assume that $u_0 = 0$. Then using the fact that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ we get from $J(u_n) \to 0$ that

$$\|\nabla u_n\|_2^2 \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx \to 0, \quad \text{as } n \to \infty. \tag{5.3.2}$$

Next we shall prove that $J(u_n) \ge 0$ for $n \in \mathbb{N}^+$ sufficiently large and this will contradict the fact that $J(u_n) = \bar{m}(c_n) < 0$ for $n \in \mathbb{N}^+$. For $p \in (1 + \frac{4}{N}, \frac{N+2}{N-2}]$ if $N \ge 3$ and $p \in (1 + \frac{4}{N}, +\infty)$ if $N = 1, 2$, by Gagliardo-Nirenberg's inequality, for some constant $C > 0$ we have

$$\int_{\mathbb{R}^N} |u_n|^{p+1} dx \le C \cdot \|\nabla u_n\|_2^{\frac{N(p-1)}{2}} \cdot c_n^{\frac{(N+2)-(N-2)p}{4}} \le C \cdot \|\nabla u_n\|_2^{\frac{N(p-1)}{2}}. \tag{5.3.3}$$

Thus it follows that

$$\begin{aligned} J(u_n) &\ge \frac{1}{2}\|\nabla u_n\|_2^2 - C \cdot \|\nabla u_n\|_2^{\frac{N(p-1)}{2}} \\ &= \|\nabla u_n\|_2^2 \left(\frac{1}{2} - C \cdot \|\nabla u_n\|_2^{\frac{Np-(N+4)}{2}} \right). \end{aligned}$$

This, together with (5.3.2), proves that $J(u_n) \ge 0$ as $n \in \mathbb{N}^+$ is sufficiently large. For $p \in (\frac{N+2}{N-2}, 3 + \frac{4}{N}), N \ge 3$, we know from the proof of [41, Theorem 1.12] that $\{u_n\}$ it is bounded in $L^q(\mathbb{R}^N)$ for all $q \geq \frac{4N}{N-2}$. Thus by Hölder and Sobolev's inequalities we can write

$$\int_{\mathbb{R}^N} |u_n|^{p+1} dx \quad \leq \quad C \cdot \|\nabla u_n\|_2^{\alpha} \cdot \|u_n\|_{(p-1)N}^{\beta}, \tag{5.3.4}$$

where

$$\alpha = \frac{2N(p-1) - 2(p+1)}{(p-1)(N-2) - 2} \quad \text{and} \quad \beta = (p-1)\frac{(N-2)(p+1) - 2N}{(p-1)(N-2) - 2}.$$

For more details see, in particular, [41, (4.16)]. Now since $\|u_n\|_{(p-1)N}^{\beta}$ is bounded we have

$$\begin{aligned} J(u_n) \quad &\geq \quad \frac{1}{2}\|\nabla u_n\|_2^2 - C \cdot \|\nabla u_n\|_2^{\alpha} \\ &= \quad \|\nabla u_n\|_2^2 \left(\frac{1}{2} - C \cdot \|\nabla u_n\|_2^{\alpha - 2}\right). \end{aligned}$$

Since $\alpha - 2 > 0$ as $p > 1$, we then deduce using (5.3.2) that $J(u_n) \geq 0$ for all $n \in \mathbb{N}^+$ sufficiently large. This proves that $u_0 \neq 0$. Finally if we assume that $\|u_0\|_2^2 < c(p, N)$ we directly get a contradiction from Lemma 5.2.1 since $\bar{m}(c) = 0$ for all $c \in (0, c(p, N)]$. Thus $\|u_0\|_2^2 = c(p, N)$ and u_0 is a minimizer of $\bar{m}(c(p, N))$. ☐

5.4 Proofs of the main results

Before the proofs of the main results, we should point out that, in the proofs of Theorems 5.1.1 and 5.1.2 we only provide the parts which were not established or whose proofs in [41] contains a gap.

Proof of Theorem 5.1.1. In [41, Theorem 1.12], Point (i) was already proved except for the statement that $\bar{m}(c(p, N)) = 0$. But it is a direct consequence of Point (ii) that we shall now prove. Let $c > 0$ be arbitrary but fixed and let $\{c_n\}$ be a sequence such that $c_n \to c$. We need to show that $\bar{m}(c_n) \to \bar{m}(c)$. By the definition of $\bar{m}(c_n)$, for each $n \in \mathbb{N}$, there exists a $u_n \in \bar{S}(c_n)$ such that

$$J(u_n) \leq \bar{m}(c_n) + \frac{1}{n}. \tag{5.4.1}$$

It is shown in [41] that $\bar{m}(c) \leq 0$ for any $c > 0$. Thus in particular

$$J(u_n) \leq \frac{1}{n}. \tag{5.4.2}$$

Now we claim that the sequences $\{\|\nabla u_n\|_2^2\}$, $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$, $\{\|u_n\|_{p+1}^{p+1}\}$ are bounded. Indeed using (5.2.3) and (5.4.2), we have

$$\frac{1}{n} \geq J(u_n) \geq \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx - \frac{C}{p+1} c_n^{1-\theta} \left(\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\right)^{\frac{\theta N}{N-2}}. \tag{5.4.3}$$

Since $\frac{\theta N}{N-2} < 1$ as $p \in [1 + \frac{4}{N}, 3 + \frac{4}{N})$, we conclude from (5.4.3) that $\{\int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx\}$ is bounded and then from (5.2.3) that $\{\|u_n\|_{p+1}^{p+1}\}$ is also bounded. At this point the fact that $\{\|\nabla u_n\|_2^2\}$ is bounded follows from the boundedness of $J(u_n)$. Now we see that

$$\bar{m}(c) \quad \leq \quad J\left(\sqrt{\frac{c}{c_n}} u_n\right)$$

$$
\begin{aligned}
&= \frac{1}{2}\left(\frac{c}{c_n}\right)\|\nabla u_n\|_2^2 + \left(\frac{c}{c_n}\right)^2 \int_{\mathbb{R}^N} |u_n|^2|\nabla u_n|^2 dx - \frac{1}{p+1}\left(\frac{c}{c_n}\right)^{\frac{p+1}{2}} \|u_n\|_{p+1}^{p+1} \\
&= J(u_n) + o(1) \le \bar{m}(c_n) + o(1).
\end{aligned}
$$

On the other hand, for a minimizing sequence $\{v_m\}$ of $\bar{m}(c)$, we have

$$
\bar{m}(c_n) \le J\left(\sqrt{\frac{c_n}{c}} v_m\right) = J(v_m) + o(1) = \bar{m}(c) + o(1).
$$

From these two estimates we deduce that $\lim_{n\to\infty} \bar{m}(c_n) = \bar{m}(c)$.

We now prove Point (iii). Note that the statement in [41, Theorem 1.12] concerning $p = 3 + \frac{4}{N}$ was incorrect. We already know, from Lemma 5.2.2, that $c_N \in (0,\infty)$. Using the definition of c_N, it follows directly that $\bar{m}(c) = 0$ for any $c \in (0, c_N)$, since one always has $\bar{m}(c) \le 0$ for any $c \in (0,\infty)$. Now if $c > c_N$, we proceed as in the proof of Theorem 2.1.1 (v), namely we observe that there exists a $v \in \bar{S}(c)$ such that $J(v) \le 0$. Indeed if we assume that $J(u) > 0$ for all $u \in \bar{S}(c)$ we reach a contradiction as follows. For an arbitrary $\hat{c} \in [c_N, c)$ taking any $u \in \bar{S}(\hat{c})$ we scale it as in (5.2.5) where $t = c/\hat{c}$. Then $u_t \in \bar{S}(c)$ and it follows from (5.2.6) that $J(u_t) \le tJ(u)$. This implies that $J(u) > 0$ for all $u \in \bar{S}(\hat{c})$ and since $\hat{c} \in [c_N, c)$ is arbitrary this contradicts the definition of $c_N > 0$.

Hence, for any $c \in (c_N,\infty)$, there exists a $u_0 \in \bar{S}(c)$ such that $J(u_0) \le 0$ and we consider the scaling

$$
u^\delta(x) = \delta^{\frac{N}{2}} u_0(\delta x), \qquad \text{for all } \delta > 0. \tag{5.4.4}
$$

Then $u^\delta \in \bar{S}(c)$, for all $\delta > 0$ and

$$
\begin{aligned}
J(u^\delta) &= \frac{\delta^2}{2}\|\nabla u_0\|_2^2 + \delta^{N+2}\int_{\mathbb{R}^N} |u_0|^2|\nabla u_0|^2 dx - \frac{N}{4(N+1)}\delta^{N+2}\|u_0\|_{4+4/N}^{4+4/N} \\
&= \frac{\delta^2}{2}\|\nabla u_0\|_2^2 - \delta^{N+2}\left(\frac{N}{4(N+1)}\|u_0\|_{4+4/N}^{4+4/N} - \int_{\mathbb{R}^N} |u_0|^2|\nabla u_0|^2 dx\right).
\end{aligned} \tag{5.4.5}
$$

Since $J(u_0) \le 0$, necessarily

$$
\frac{N}{4(N+1)}\|u_0\|_{4+4/N}^{4+4/N} - \int_{\mathbb{R}^N} |u_0|^2|\nabla u_0|^2 dx > 0
$$

and thus we see from (5.4.5) that $\lim_{\delta\to\infty} J(u^\delta) = -\infty$. It proves that $\bar{m}(c) = -\infty$ for any $c \in (c_N, +\infty)$. □

Proof of Theorem 5.1.2. In [41, Theorem 1.12] it is shown that $\bar{m}(c)$ admits a minimizer if $c \in (c(p,N),\infty)$. By Lemma 5.3.1 this is also true for $c = c(p,N)$. To complete the proof of Point (i) we need to show that for $c \in (0, c(p,N))$, $\bar{m}(c)$ does not admit a minimizer. But since $\bar{m}(c) = 0$ for $c \in (0, c(p,N)]$ it results directly from Lemma 5.2.1. To prove Point (ii) we argue by contradiction assuming that there exists a $c > 0$ such that $\bar{m}(c)$ admits a minimizer u_c. Then, by standard arguments, u_c satisfies weakly

$$
-\Delta u_c - \lambda_c u_c - u_c\Delta|u_c|^2 = |u_c|^{p-1}u_c, \tag{5.4.6}
$$

where $\lambda_c \in \mathbb{R}$ is the associated Lagrange multiplier. Multiplying (5.4.6) by u_c and integrating we derive that

$$
\int_{\mathbb{R}^N} |\nabla u_c|^2 dx + 4\int_{\mathbb{R}^N} |u_c|^2|\nabla u_c|^2 dx - \int_{\mathbb{R}^N} |u_c|^{p+1} dx = \lambda_c\|u_c\|_2^2. \tag{5.4.7}
$$

Also, from [41, Lemma 3.1] we know that u_c satisfies the Pohozaev identity

$$\frac{N-2}{N}\left(\frac{1}{2}\int_{\mathbb{R}^N}|\nabla u_c|^2dx+\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx\right)=\frac{\lambda_c}{2}\|u_c\|_2^2+\frac{1}{p+1}\|u_c\|_{p+1}^{p+1}. \tag{5.4.8}$$

It follows from (5.4.7) and (5.4.8) that

$$\|\nabla u_c\|_2^2+(N+2)\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx-\frac{N(p-1)}{2(p+1)}\|u_c\|_{p+1}^{p+1}=0, \tag{5.4.9}$$

by which we can rewrite $J(u_c)$ as

$$J(u_c)=\frac{Np-(N+4)}{2N(p-1)}\|\nabla u_c\|_2^2+\frac{Np-(3N+4)}{N(p-1)}\int_{\mathbb{R}^N}|u_c|^2|\nabla u_c|^2dx. \tag{5.4.10}$$

When $p=3+\frac{4}{N}$, (5.4.10) becomes

$$J(u_c)=\frac{N}{2N+4}\|\nabla u_c\|_2^2. \tag{5.4.11}$$

This is clearly a contradiction since by assumption $J(u_c)=\bar{m}(c)\leq 0$ and Point (ii) is established. □

Proof of Theorem 5.1.6. From the proof of Theorem 5.1.2, we know that any critical point u_c of J restricted to $\bar{S}(c)$ must satisfy (5.4.9). Denoting

$$\bar{Q}(u)=\|\nabla u\|_2^2+(N+2)\int_{\mathbb{R}^N}|u|^2|\nabla u|^2dx-\frac{N(p-1)}{2(p+1)}\|u\|_{p+1}^{p+1},$$

we thus have $\bar{Q}(u_c)=0$. Now we assume by contradiction that there exist sequence $\{c_n\}\subset\mathbb{R}^+$ with $c_n\to 0$, and $\{u_n\}\subset\bar{S}(c_n)$ such that u_n is a critical point of J on $\bar{S}(c_n)$. Then for each $n\in\mathbb{N}$, $\bar{Q}(u_n)=0$ and using (5.2.3) we obtain

$$\|\nabla u_n\|_2^2+(N+2)\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\leq C\cdot c_n^{1-\theta}\cdot\left(\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\right)^{\frac{\theta N}{N-2}}, \tag{5.4.12}$$

where $\theta=\frac{(p-1)(N-2)}{2(N+2)}$. When $p=3+\frac{4}{N}$ we have $\frac{\theta N}{N-2}=1$, $1-\theta=\frac{4}{N}$ and thus we get immediately a contradiction from (5.4.12). Now when $p\in[1+\frac{4}{N},3+\frac{4}{N})$, $\frac{\theta N}{N-2}<1$ and we derive from (5.4.12) that

$$\int_{\mathbb{R}^N}|u_n|^2|\nabla u_n|^2dx\to 0\ \text{ and } \|\nabla u_n\|_2^2\to 0\ \text{ as } n\to\infty. \tag{5.4.13}$$

Also when $p\in[1+\frac{4}{N},\frac{N+2}{N-2}]$ if $N\geq 3$ and $p\in[1+\frac{4}{N},+\infty)$ if $N=1,2$, we obtain from (5.3.3) that

$$\begin{aligned}\bar{Q}(u_n) &\geq \|\nabla u_n\|_2^2-C\cdot\|\nabla u_n\|_2^{\frac{N(p-1)}{2}}\cdot c_n^{\frac{(N+2)-(N-2)p}{4}}\\ &= \|\nabla u_n\|_2^2\left(1-C\cdot\|\nabla u_n\|_2^{\frac{Np-(N+4)}{2}}\cdot c_n^{\frac{(N+2)-(N-2)p}{4}}\right).\end{aligned} \tag{5.4.14}$$

Taking (5.4.13) into account (5.4.14) implies that $\bar{Q}(u_n)>0$ for $n\in\mathbb{N}$ large enough and provides a contradiction.

When $p\in(\frac{N+2}{N-2},3+\frac{4}{N})$, $N\geq 3$, using (5.3.4) and the fact that $\{\|u_n\|_{(p-1)N}^{\beta}\}$ is bounded, we have

$$\bar{Q}(u_n)\ \geq\ \|\nabla u_n\|_2^2-C\cdot\|\nabla u_n\|_2^{\alpha}.$$

Since $\alpha-2$ as $p>1$, using (5.4.13) we conclude that $\bar{Q}(u_n)>0$ for $n\in\mathbb{N}$ sufficiently large. Here also we have obtained a contradiction and this ends the proof. □

Chapter 6

Multiple normalized solutions for quasi-linear Schrödinger equations

6.1 Introduction

In this chapter, we are concerned with quasi-linear Schrödinger equations of the form

$$\begin{cases} i\partial_t\varphi + \Delta\varphi + \varphi\Delta|\varphi|^2 + |\varphi|^{p-1}\varphi = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ \varphi(0,x) = \varphi_0(x), & \text{in } \mathbb{R}^N, \end{cases} \tag{6.1.1}$$

where $p \in (1, \frac{3N+2}{N-2})$ if $N \geq 3$ and $p \in (1,\infty)$ if $N = 1, 2$, and the function $\varphi : \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{C}$ is complex valued. Such types of equations are used in various physical fields. For example, in dissipative quantum mechanics, in plasma physics, in fluid mechanics. We refer to the Introduction chapter or [41] and its references for more information about the physical backgrounds.

From the mathematical point of view, a strong focus is made on the existence and dynamics of standing waves of (6.1.1). By standing waves, we mean solutions of the form $\varphi(t,x) = e^{-i\lambda t}u(x)$, where $\lambda \in \mathbb{R}$ is a parameter. Observe that $e^{-i\lambda t}u(x)$ solves (6.1.1) if and only if $u(x)$ satisfies the following stationary equation

$$-\Delta u - u\Delta(u^2) - \lambda u - |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^N. \tag{P_λ}$$

In (P_λ), when $\lambda \in \mathbb{R}$ is a fixed parameter, the existence and multiplicity of solutions of (P_λ) have been intensively studied during the last decade. See [5, 40, 41, 50, 86, 87, 88, 89, 97, 101] and their references therein. We also refer to the works [1, 5, 54, 103] for the uniqueness of ground states of (P_λ). Ground states here mean solutions of (P_λ) which minimize among all nontrivial solutions of (P_λ) the associated energy functional

$$I_\lambda(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{\lambda}{2}\int_{\mathbb{R}^N} |u|^2 dx + \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^N} |u|^{p+1} dx,$$

defined on the natural space

$$\mathcal{X} := \Big\{u \in W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx < \infty\Big\}.$$

It is known from [89] or [101] that when $N \geq 3$ the exponent $\frac{3N+2}{N-2}$ acts as a critical exponent of (P_λ). See [89] or [101], and also Remark 1.2.1, for more details.

In the present chapter motivated by the fact that physicists are often interested in "normalized solutions" we look for solutions to (P_λ) having a prescribed L^2-norm. In this aim, for given $c > 0$, one can look to minimizers of

$$\bar{m}(c) := \inf_{u \in \bar{S}(c)} J(u), \tag{6.1.2}$$

where

$$\bar{S}(c) := \Big\{ u \in \mathcal{X} : \int_{\mathbb{R}^N} |u|^2 dx = c \Big\}. \tag{6.1.3}$$

Here the functional $J : \bar{S}(c) \to \mathbb{R}$, is defined as

$$J(u) := \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx - \frac{1}{p+1} \int_{\mathbb{R}^N} |u|^{p+1} dx. \tag{6.1.4}$$

It is known from [41, Theorem 4.6] that for each minimizer $u \in \bar{S}(c)$ of (6.1.2), there exists a Lagrange parameter $\lambda < 0$ such that the couple (u, λ) solves (P_λ).

We collect below the known results concerning minimizers of $\bar{m}(c)$.

Lemma 6.1.1. *([41, Theorems 1.9, 1.12], [65, Theorems 1.4, 1.5]) Assume that* $p \in (1, \frac{3N+2}{N-2})$ *if* $N \geq 3$ *and* $p \in (1, \infty)$ *if* $N = 1, 2$. *Then*

(1) Concerning the properties of the function $c \to \bar{m}(c)$, *we have*

   i) For all $c > 0$, $\bar{m}(c) \in (-\infty, 0]$ *as* $p \in (1, 3 + \frac{4}{N})$;

   ii) For all $c > 0$, $\bar{m}(c) = -\infty$ *as* $p \in (3 + \frac{4}{N}, \frac{3N+2}{N-2})$ *if* $N \geq 3$ *and* $p \in (3 + \frac{4}{N}, \infty)$ *if* $N = 1, 2$;

   iii) For $p = 3 + \frac{4}{N}$, *there exists a* $c_N > 0$, *given by*

$$c_N := \inf\{c > 0 : \ \exists\ u \in \bar{S}(c) \ \textit{such that}\ J(u) \leq 0\},$$

   such that

$$\begin{cases} \bar{m}(c) = 0, & \textit{as}\ c \in (0,\ c_N); \\ \bar{m}(c) = -\infty, & \textit{as}\ c \in (c_N, \infty). \end{cases}$$

(2) When $p \in (1, 1 + \frac{4}{N})$, *for all* $c > 0$, $\bar{m}(c) < 0$ *and* $\bar{m}(c)$ *has a minimizer.*

(3) When $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$, *there exists a* $c(p, N) > 0$, *given by*

$$c(p, N) := \inf\{c > 0 : \ \bar{m}(c) < 0\}, \tag{6.1.5}$$

such that

   i) If $c \in (0, c(p, N))$, $\bar{m}(c) = 0$ *and* $\bar{m}(c)$ *has no minimizer;*

   ii) If $c = c(p, N)$, $\bar{m}(c) = 0$ *and* $\bar{m}(c)$ *admits a minimizer;*

   iii) If $c \in (c(p, N), \infty)$, $\bar{m}(c) < 0$ *and* $\bar{m}(c)$ *admits a minimizer.*

(4) When $p = 3 + \frac{4}{N}$, *for all* $c > 0$, $\bar{m}(c)$ *admits no minimizers.*

(5) The standing waves obtained as minimizers of $\bar{m}(c)$ *are orbitally stable.*

In this chapter, we are interested in the range $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$, $N \geq 1$. From Lemma 6.1.1 (3), we know that in this range the functional J has, for $c \geq c(p,N)$, a critical point on $\bar{S}(c)$, which is a global minimizer. Here we extend this result in two directions. First we prove that there exists a $c_0 \in (0, c(p,N))$ such that, for each $c \in (c_0, c(p,N))$ the functional J admits on $\bar{S}(c)$ a local minimizer. By Lemma 6.1.1 (3) i), this local minimizer can not be a global one. Secondly we show that when $c \in (c_0, \infty)$ the functional J admits on $\bar{S}(c)$ a second critical point of mountain pass type. Note that since J is not differentiable we must give a meaning to what we call a critical point of J on $\bar{S}(c)$. By definition it will be a solution of (P_λ) belonging to $\bar{S}(c)$.

The main result of this chapter is the following theorem.

Theorem 6.1.2. *Assume that $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$ if $N = 1,2,3$ and $p \in (1+\frac{4}{N}, \frac{N+2}{N-2}]$ if $N \geq 4$. Then there exists a $c_0 \in (0, c(p,N))$ such that for any $c \in (c_0, \infty)$ the functional J admits two critical points u_c and v_c on $\bar{S}(c)$. In addition*

(1) $J(u_c) > J(v_c)$ for any $c \in (c_0, \infty)$.

(2) $J(u_c) > 0$ for all $c \in (c_0, \infty)$ and $J(u_c)$ is a mountain pass level.

(3) $$J(v_c) \begin{cases} > 0, & \text{if } c \in (c_0, c(p,N)); \\ = 0, & \text{if } c = c(p,N); \\ < 0, & \text{if } c \in (c(p,N), \infty). \end{cases}$$

Also v_c is a local minimum of J when $c \in (c_0, c(p,N))$ and a global minimum of J when $c \in [c(p,N), \infty)$.

(4) u_c and v_c are Schwarz symmetric functions.

(5) There exists Lagrange multipliers $\lambda_c < 0$ and $\beta_c < 0$ such that (u_c, λ_c) and (v_c, β_c) solve (P_λ).

Remark 6.1.3. In Theorem 5.1.6 it is proved that there exists a $\hat{c} > 0$ such that for all $c \in (0, \hat{c})$ the functional J restricted to $\bar{S}(c)$ has no critical point. It is an open question whether or not we can take $c_0 = \hat{c}$ in Theorem 6.1.2. Already it would be interesting to know if the set of $c \in (0, c(p,N)]$ for which one can find the two critical points u_c and v_c of Theorem 6.1.2 is an interval.

In addition, in the case $c \in [c(p,N), \infty)$ the critical point v_c is just a global minimizer already obtained in [41, 65] whose existence is recalled in Lemma 6.1.1.

As we mentioned in the Introduction chapter the functional J is not differentiable except when $N = 1$. To overcome the lack of differentiability of J we apply a perturbation method recently developed in [86]. That is, we consider first the perturbed functional

$$J_\mu(u) := \frac{\mu}{4} \int_{\mathbb{R}^N} |\nabla u|^4 dx + J(u), \tag{6.1.6}$$

where $\mu \in (0,1]$ is a parameter. For any given $c > 0$, we denote

$$\Sigma_c := \Big\{ u \in W^{1,4} \cap W^{1,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 dx = c \Big\}.$$

One may observe that $J_\mu(u)$ is well-defined and C^1 in Σ_c (see [86]).

The idea is to look to critical points of J_μ, for each $\mu > 0$ small and then, having obtained these critical points, to show that they converge to suitable critical points of J.

A first critical point u_μ^c of J_μ is obtained at a critical value $\gamma_\mu(c) > 0$ which corresponds to a mountain pass level. When $c \in (c_0, c(p, N))$ a second critical point v_μ^c is obtained as a local minimizer of J_μ. The corresponding energy level $\tilde{m}_\mu(c)$ is strictly positive. To derive these results we first check the geometric properties of J_μ allowing to search for such critical points. To show that these critical levels are actually reached, several difficulties have to be overcome. Since J_μ is coercive on Σ_c any Palais-Smale sequence $\{u_n\} \subset \Sigma_c$ is bounded and thus we can assume that $u_n \rightharpoonup u_c$. It is also standard to show that there exists a $\lambda_c \in \mathbb{R}$ such that $J'_\mu(u_c) - \lambda_c u_c = 0$. Finally we mention that, by constructing Palais-Smale sequences which consist of almost Schwarz symmetric functions we can avoid any problems related to possible dichotomy of our sequences, in the sense of P. L. Lions [83]. The first main difficulty is to show that $u_c \neq 0$. To overcome it we need, for both $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$, to establish the existence of Palais-Smale sequences having the additional property that $Q_\mu(u_n) \to 0$. See Lemmas 6.2.6 and 6.2.9.

In the case of $\gamma_\mu(c)$ the existence of such Palais-Smale sequence is proved using the trick, first introduced in [64], to construct an auxiliary functional on $\Sigma_c \times \mathbb{R}$. This trick, which has been used recently on various problems [8, 59, 95] permits to incorporate in the variational procedure the information that any critical point of J_μ on $\Sigma(c)$ must satisfy $Q_\mu(u) = 0$. For $\tilde{m}_\mu(c)$ we can *directly* construct a minimizing sequence $\{u_n\} \subset \Sigma(c)$ satisfying $Q_\mu(u_n) = 0, \forall n \in \mathbb{N}$. It readily leads to the fact that the weak limit of the associated Palais-Smale sequence is non trivial.

Another main difficulty is to show that the weak limit u_c does belong to Σ_c, namely that $||u_c||_2^2 = c$. For this we need to require that $\lambda_c \in \mathbb{R}$ satisfies $\lambda_c < 0$. Here, and only here, comes the need to restrict our result from the *natural* range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$ for any $N \geq 1$ to the range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$ when $N = 1, 2, 3$ and $(1 + \frac{4}{N}, \frac{N+2}{N-2})$ when $N \geq 4$. It is not clear to us if it is possible to prove that $\lambda_c < 0$ for our critical points in all the range $(1 + \frac{4}{N}, 3 + \frac{4}{N})$. Also we do not know if $\lambda_c < 0$ is necessary to get a critical point on Σ_c. However let us mention that in [16] we faced a similar issue but there strong indications incline to think it is necessary for the suspected Lagrange multipliers to be strictly negative.

Having proved the existence of the critical points u_μ^c and v_μ^c at the levels $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$ respectively we pass to the limit $\mu \to 0$ and we show that $u_\mu^c \to u_c$ and $v_\mu(c) \to v_c$ where u_c and v_c are as presented in Theorem 6.1.2. In this part we strongly rely on the machinery developed in [86].

In this chapter we also discuss the behavior of the Lagrange multipliers corresponding to the global minimizers of J.

Lemma 6.1.4. *Assume that $p \in (1, 3+\frac{4}{N})$ and for $c > 0$ large, let v_c be a global minimizer of J on $S(c)$ and $\beta_c < 0$ be its Lagrange multiplier. Then*

$$\beta_c \to -\infty \quad as \quad c \to \infty.$$

Finally, we present a relationship between the ground states of (P_λ) and the global minimizers of $\bar{m}(c)$.

Theorem 6.1.5. *Assume that $p \in (1, 3 + \frac{4}{N})$ and for some $c > 0$ let $u_c \in \bar{S}(c)$ be a global minimizer of $m(c)$ and $\beta_c < 0$ be its Lagrange multiplier. Then u_c is a ground state solution of (P_λ) with $\lambda = \beta_c$.*

Note however that the converse of Theorem 6.1.5 does not hold in general. Indeed on one hand our mountain pass solution is non negative. On the other hand we know in several cases that (P_λ) has a unique non negative solution when $\lambda > 0$ is fixed. This is the case in particular when $N = 1$, see [41, Theorem 1.3] or [5]. Thus when this uniqueness property holds our mountain pass solution must necessarily be a ground state. This observation shows that not all ground states of (P_λ) for $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ can be obtained as minimizers of J on the corresponding constraint.

Remark 6.1.6. From Theorem 6.1.5 we deduce that any global minimizer u_c has a given sign and that $|u_c|$ is a radially symmetric, decreasing function with respect to one point. This follows directly from [41, Theorem 1.3].

6.2 Perturbation of the functional

In this section to overcome the non-differentiability of the functional J, we apply the perturbation method introduced in [86]. First, we show that there exists a $c_0 \in (0, c(p, N))$ such that, for each $c \in (c_0, \infty)$ the functional J_μ has a mountain pass geometry on Σ_c when $\mu > 0$ is sufficiently small. For simplicity, we denote in this section $X := W^{1,4} \cap W^{1,2}(\mathbb{R}^N)$.

Lemma 6.2.1. *[Mountain Pass geometry] Assume that $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N}), N \geq 1$. Then there exists a $c_0 \in (0, c(p, N))$ such that for any fixed $c \in [c_0, \infty)$ taking $\mu_0 > 0$ small enough the functional J_μ has, for $\mu \in (0, \mu_0)$ a Mountain Pass geometry on the constraint Σ_c. Precisely there exist $(u_0, u_1) \in \Sigma_c \times \Sigma_c$ both Schwarz symmetric, such that*

$$\gamma_\mu(c) = \inf_{g \in \Gamma_c} \max_{t \in [0,1]} J_\mu(g(t)) > \max\{J_\mu(u_0), J_\mu(u_1)\} > 0 \tag{6.2.1}$$

where

$$\Gamma_c = \{g \in C([0, 1], \Sigma_c) : \ g(0) = u_0, \ g(1) = u_1\}.$$

The proof of Lemma 6.2.1 relies on the following estimates.

Lemma 6.2.2. *Assume that $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$, $N \geq 1$. Then setting for $k > 0$,*

$$C_k := \{u \in \Sigma_c : \int_{\mathbb{R}^N} (1 + |u|^2)|\nabla u|^2 dx = k\}$$

there exists a $k_0 > 0$ sufficiently small such that for all $k \in (0, k_0]$ and all $\mu > 0$

$$J_\mu(u) \geq \frac{1}{4} k > 0, \quad \text{for all } u \in C_k. \tag{6.2.2}$$

$$Q_\mu(u) \geq \frac{1}{4} k > 0, \quad \text{for all } u \in C_k. \tag{6.2.3}$$

Here we have set, for any given $\mu > 0$,

$$Q_\mu(u) = \frac{\mu(N + 4)}{4} ||\nabla u||_4^4 + ||\nabla u||_2^2 + (N + 2) \int_{\mathbb{R}^N} |u|^2 |\nabla u|^2 dx - \frac{N(p - 1)}{2(p + 1)} ||u||_{p+1}^{p+1}. \tag{6.2.4}$$

Moreover when $c \in (0, c(p, N)]$, the constant $k_0 > 0$ can be chosen independently of $c > 0$.

Proof. When $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ if $N = 1, 2, 3$ and $p \in (1 + \frac{4}{N}, \frac{N+2}{N-2})$ if $N \geq 4$, by Gagliardo-Nirenberg's inequality, there exists a $C = C(p, N) > 0$, such that for all $u \in X$,

$$\int_{\mathbb{R}^N} |u|^{p+1} dx \leq C \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{N(p-1)}{4}} \left(\int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{(N+2)-(N-2)p}{4}}$$

$$\leq C\left(\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\right)^{\frac{N(p-1)}{4}}\left(\int_{\mathbb{R}^N}|u|^2dx\right)^{\frac{(N+2)-(N-2)p}{4}} \tag{6.2.5}$$

Thus, for a constant $C=C(p,N)>0$, independent of $c>0$ when $c\in(0,c(p,N))$, we have for all $u\in\Sigma_c$

$$J_\mu(u)\geq\frac{1}{2}\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx-C\left(\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\right)^{\frac{N(p-1)}{4}}. \tag{6.2.6}$$

when $p\in(1+\frac{4}{N},3+\frac{4}{N})$ if $N=1,2,3$ and $p\in(1+\frac{4}{N},\frac{N+2}{N-2})$ if $N\geq4$.

When $p\in[\frac{N+2}{N-2},3+\frac{4}{N}),N\geq4$, we claim that there exists a $C=C(p,N)>0$, such that for all $u\in X$,

$$\int_{\mathbb{R}^N}|u|^{p+1}dx\leq C\left(\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\right)^{\frac{N}{N-2}}. \tag{6.2.7}$$

To show (6.2.7), let $\alpha\in\mathbb{R}$ be such that

$$\frac{p+1}{\alpha+1}=\frac{2N}{N-2}.$$

Clearly, $\alpha\in[0,1]$ if $p\in[\frac{N+2}{N-2},3+\frac{4}{N}),N\geq4$. Then by Sobolev-Gagliardo-Nirenberg's inequality [23, Theorem IX.9], there exists a $C=C(p,N)>0$ such that for all $u\in X$,

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^{p+1}dx &= \int_{\mathbb{R}^N}(|u|^{\alpha+1})^{\frac{2N}{N-2}}dx\leq C\left(\int_{\mathbb{R}^N}|\nabla(|u|^{\alpha+1})|^2dx\right)^{\frac{N}{N-2}}\\ &= C\left(\int_{\mathbb{R}^N}|u|^{2\alpha}|\nabla u|^2dx\right)^{\frac{N}{N-2}}.\end{aligned}$$

Since $\alpha\in[0,1]$, from the last inequality we conclude that (6.2.7) holds. Then

$$J_\mu(u)\geq\frac{1}{2}\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx-C\left(\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\right)^{\frac{N}{N-2}}, \tag{6.2.8}$$

when $p\in[\frac{N+2}{N-2},3+\frac{4}{N}),N\geq4$.

We note that $\frac{N(p-1)}{4}>1$ as $p>1+\frac{4}{N}$ and $\frac{N}{N-2}>1$ as $N\geq4$. Thus when $p\in(1+\frac{4}{N},3+\frac{4}{N})$, $N\geq1$, by (6.2.6)-(6.2.8), there exists a $k_0>0$ small, such that for all $k\in(0,k_0)$,

$$J_\mu(u)\geq\frac{1}{4}k>0,\quad\text{for all } u\in C_k.$$

Note that when $c\in(0,c(p,N)]$ the constant $k_0>0$ can be chosen independently of $c>0$. This proves that (6.2.2) holds. Now by the estimates (6.2.6) and (6.2.8), one obtain the existence of a constant $C=C(p,N)>0$, independent of $u\in X$, such that for all $u\in\Sigma_c$,

$$Q_\mu(u)\geq\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx-C\Big\{\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\Big\}^{\frac{N(p-1)}{4}}, \tag{6.2.9}$$

where $p\in(1+\frac{4}{N},3+\frac{4}{N})$ if $N=1,2,3$ and $p\in(1+\frac{4}{N},\frac{N+2}{N-2})$ if $N\geq4$, and

$$Q_\mu(u)\geq\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx-C\Big\{\int_{\mathbb{R}^N}(1+|u|^2)|\nabla u|^2dx\Big\}^{\frac{N}{N-2}}, \tag{6.2.10}$$

where $p\in[\frac{N+2}{N-2},3+\frac{4}{N})$ as $N\geq4$. From (6.2.9) and (6.2.10) we also readily derive that (6.2.3) holds. $\square$

Proof of Lemma 6.2.1. We know from Lemma 6.1.1 (3) b) that J has, when $c \geq c(p,N)$, a global minimizer $u_c \in W^{1,2}(\mathbb{R}^N)$. By density we can replace u_c by $\tilde{u}_c \in X$ with $J(\tilde{u}_c)$ arbitrarily close to $J(u_c) = m(c)$. When $c > c(p,N)$ we have $m(c) < 0$ and thus taking $\mu_0 > 0$ small enough we have that $J_\mu(\tilde{u}_c) < 0$ for all $\mu \in (0, \mu_0)$. We set $u_1 = \tilde{u}_c^*$, where $\tilde{u}_c^*$ denotes the Schwarz symmetrization of $\tilde{u}_c$. One notes that by the Polya-Szegö inequality $\|\nabla u^*\|_q^q \leq \|\nabla u\|_q^q$, $\forall q \in [1,\infty)$, and using [41, Lemma 4.3], we have $J_\mu(\tilde{u}_c^*) \leq J_\mu(\tilde{u}_c)$. Also clearly taking $k_0 > 0$ smaller if necessary we have that $\int_{\mathbb{R}^N}(1+|u_1|^2)|\nabla u_1|^2 dx > k_0$.

Now let $t < 1$ be close to 1 and set $c_0 = tc(p,N)$. By the continuity of J_μ, taking $t < 1$ sufficiently close to 1 we have that

$$J_\mu(\sqrt{t}\tilde{u}_c) < \frac{1}{4}k_0.$$

We fix such a $t < 1$ and we set $u_1 = \sqrt{t}\tilde{u}_c^*$. Without restriction we can also assume that $\int_{\mathbb{R}^N}(1+|u_1|^2)|\nabla u_1|^2 dx > k_0$.

To choose $u_0 \in \Sigma_c$, we consider the scaling

$$u_1^\theta(x) := \theta^{N/2} u_1(\theta x), \ \forall\, \theta > 0.$$

By direct calculations we observe that

1) $u_1^\theta \in \Sigma_c$, $\forall\, \theta > 0$;

2) $\lim_{\theta\to 0} J_\mu(u_1^\theta) = 0$;

3) $\lim_{\theta\to 0} \int_{\mathbb{R}^N}(1+|u_1^\theta|^2)|\nabla u_1^\theta|^2 dx = 0$.

Thus there exists $\theta_0 > 0$ sufficiently small such that

$$J_\mu(u_1^{\theta_0}) \leq \frac{1}{8}k_0, \ \text{and} \ \int_{\mathbb{R}^N}(1+|u_1^{\theta_0}|^2)|\nabla u_1^{\theta_0}|^2 dx < k_0.$$

Letting $u_0 = u_1^{\theta_0}$, this and Lemma 6.2.2 complete the proof. □

Remark 6.2.3. By choosing $u_0, u_1 \in \Sigma_c$ both being Schwarz symmetric, one has that g^* is still a path in Γ_c if $g \in \Gamma_c$, which will be used in the following Lemma.

Also one observes that the inequality (6.2.7) holds for any $p \in [\frac{N+2}{N-2}, \frac{3N+2}{N-2}]$ if $N \geq 3$. Accordingly the mountain pass geometry of Lemma 6.2.1 still holds in that range.

We now show that the geometry of J presents a local minimum when $c \in (c_0, c(p,N))$. Actually we shall get a local minimizer of J_μ on Σ_c by considering the minimization problem

$$\tilde{m}_\mu(c) := \inf_{u\in\Sigma_c\setminus B} J_\mu(u), \tag{6.2.11}$$

where $B := \bigcup_{0<k\leq k_0} C_k$ and $k_0 > 0$ is given in Lemma 6.2.2.

Lemma 6.2.4 (Geometry of local minima)**.** *For any given $c \in (c_0, c(p,N))$ we have*

$$0 \leq \tilde{m}_\mu(c) = \inf_{u\in\Sigma_c\setminus B} J_\mu(u) < \inf_{u\in C_{k_0}} J_\mu(u). \tag{6.2.12}$$

Proof. From the proof of Lemma 6.2.1, we know that there exists a $v_0 \in \Sigma_c \setminus B$ such that $J_\mu(v_0) < \inf_{u \in C_{k_0}} J_\mu(u)$. Thus

$$\inf_{u\in\Sigma_c\setminus B} J_\mu(u) \le J_\mu(v_0) < \inf_{u\in C_{k_0}} J_\mu(u).$$

In addition, by Lemma 6.1.1, we know that $\tilde{m}_\mu(c) = \inf_{u\in\Sigma_c\setminus B} J_\mu(u) \ge 0$. This completes the proof. ☐

In view of Lemmas 6.2.1 and 6.2.4 we shall search for critical points of J_μ at the levels $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$. For this we establish the existence of special Palais-Smale sequences associated with $\gamma_\mu(c)$ and $\tilde{m}_\mu(c)$ and show their convergence. In that direction we first observe

Lemma 6.2.5. *For any fixed $c \in (0,\infty)$ and any fixed $\mu > 0$, if a sequence $\{u_n\} \subset \Sigma_c$ is such that $\{J_\mu(u_n)\} \subset \mathbb{R}$ is bounded then it is bounded in X.*

Proof. From (1.2.7), we know that there holds for any $u \in X$ that,

$$\int_{\mathbb{R}^N} |u|^{q+1}dx \le C\cdot \|u\|_2^{\frac{3N+2-(N-2)q}{N+2}} \cdot \left(\int_{\mathbb{R}^N} |u|^2|\nabla u|^2 dx\right)^{\frac{N(q-1)}{2(N+2)}}, \tag{6.2.13}$$

for $q \in (1, \frac{3N+2}{N-2})$ if $N \ge 3$ and $q \in (1,\infty)$ if $N = 1, 2$. Thus we have

$$J_\mu(u_n) \ge \frac{\mu}{4}\|\nabla u_n\|_4^4 + \frac{1}{2}\int_{\mathbb{R}^N}(1+|u_n|^2)|\nabla u_n|^2dx - C\cdot\left(\int_{\mathbb{R}^N}(1+|u_n|^2)|\nabla u_n|^2dx\right)^{\frac{N(p-1)}{2(N+2)}}.$$

Since $\frac{N(p-1)}{2(N+2)} < 1$ as $p < 3 + \frac{4}{N}$, the last inequality implies from the boundedness of $\{J_\mu(u_n)\}$ that $\{\int_{\mathbb{R}^N}(1+|u_n|^2)|\nabla u_n|^2dx\}$ is bounded, and also for fixed $\mu > 0$, $\{\|\nabla u_n\|_4^4\}$ is bounded. In addition, because of (6.2.13), $\{\|u_n\|_4^4\}$ is bounded. Thus $\{u_n\}$ is bounded in X. ☐

From Lemma 6.2.5 we know, in particular, that any Palais-Smale sequences for J_μ are bounded. The need to use special Palais-Smale sequences comes from the difficulty to pass from weak convergence to strong convergence. A problem linked to possible loss of compactness, due to the fact that our equation is set on all $\mathbb{R}^N$. In order to regain some compactness we could proceed as in [86, Lemma 2.1] by working in the subspace of radially symmetry functions $W_r^{1,4} \cap W_r^{1,2}(\mathbb{R}^N)$, $N \ge 2$. However, when $N = 1$, the inclusion $H_r^1(\mathbb{R}) \subset L^q(\mathbb{R})$ for $q > 2$ is not compact and another proof is needed. Here we choose to construct special Palais-Smale sequences for J_μ which consist of almost Schwarz symmetric functions. This allows us to give a proof in any dimension. By working with such Palais-Smale sequences we avoid any problem related to possible dichotomy, in the sense of P. L. Lions [83], of our sequences.

Even if we work with sequences of almost symmetric functions it is not automatic that they converge to a non trivial weak limit. To avoid this possibility we construct Palais-Smale sequences $\{u_n\} \subset \Sigma_c$ which satisfy in addition the property that $Q_\mu(u_n) \to 0$. For the mountain pass level $\gamma_\mu(c)$ this is done using the trick introduced in [64] and already used in Chapter 4. For $\tilde{m}_\mu(c)$ a direct argument will provide the result.

Lemma 6.2.6. *[A special Palais-Smale sequence for $\gamma_\mu(c)$] Assume that $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$, $N \geq 1$. Then, for any given $c \geq c_0$ where $c_0 > 0$ is given in Lemma 6.2.1, there exist a sequence $\{u_n\} \subset \Sigma_c$ and a sequence $\{v_n\} \subset X$ of Schwarz symmetric functions, such that*

$$\begin{cases} J_\mu(u_n) \to \gamma_\mu(c) > 0, \\ \|J'_\mu|_{\Sigma_c}(u_n)\|_{X^*} \to 0, \\ Q_\mu(u_n) \to 0, \\ \|u_n - v_n\|_X \to 0, \end{cases} \tag{6.2.14}$$

as $n \to \infty$. Here X^* denotes the dual space of X.

Before proving Lemma 6.2.6 we need to introduce some notations and to prove some preliminary results.

For any fixed $\mu > 0$, we introduce the following auxiliary functional

$$\widetilde{J}_\mu : \Sigma_c \times \mathbb{R} \to \mathbb{R}, \qquad (u,s) \mapsto J_\mu(H(u,s)),$$

where $H(u(x),s) := e^{\frac{N}{2}s}u(e^s x)$, and the set of paths

$$\widetilde{\Gamma}_c := \Big\{\widetilde{g} \in C([0,1], \Sigma_c \times \mathbb{R}) : \ \widetilde{g}(0) = (u_0, 0), \ \widetilde{g}(1) = (u_1, 0)\Big\},$$

where $u_0, u_1 \in \Sigma_c$ are given in Lemma 6.2.1.

Observe that defining

$$\widetilde{\gamma}_\mu(c) := \inf_{\widetilde{g} \in \widetilde{\Gamma}_c} \max_{t \in [0,1]} \widetilde{J}_\mu(\widetilde{g}(t)),$$

we have that

$$\widetilde{\gamma}_\mu(c) = \gamma_\mu(c). \tag{6.2.15}$$

Indeed, by the definitions of $\widetilde{\gamma}_\mu(c)$ and $\gamma_\mu(c)$, (6.2.15) follows immediately from the fact that the maps

$$\varphi : \Gamma_c \longrightarrow \widetilde{\Gamma}_c, \ g \longmapsto \varphi(g) := (g, 0),$$

and

$$\psi : \widetilde{\Gamma}_c \longrightarrow \Gamma_c, \ \widetilde{g} \longmapsto \psi(\widetilde{g}) := H \circ \widetilde{g},$$

satisfy

$$\widetilde{J}_\mu(\varphi(g)) = J_\mu(g) \ \text{ and } \ J_\mu(\psi(\widetilde{g})) = \widetilde{J}_\mu(\widetilde{g}).$$

In the proof of Lemma 6.2.6, the lemma below which has been established by Ekeland's variational principle in [64, Lemma 2.3] is used. Hereinafter we denote by E the set $X \times \mathbb{R}$ equipped with the norm $\|\cdot\|_E^2 = \|\cdot\|_X^2 + |\cdot|_{\mathbb{R}}^2$ and denote by E^* its dual space.

Lemma 6.2.7. *Let $\varepsilon > 0$. Suppose that $\widetilde{g}_0 \in \widetilde{\Gamma}_c$ satisfies*

$$\max_{t \in [0,1]} \widetilde{J}_\mu(\widetilde{g}_0(t)) \leq \widetilde{\gamma}_\mu(c) + \varepsilon.$$

Then there exists a pair of $(u_0, s_0) \in \Sigma_c \times \mathbb{R}$ such that:

(1) $\widetilde{J}_\mu(u_0, s_0) \in [\widetilde{\gamma}_\mu(c) - \varepsilon, \widetilde{\gamma}_\mu(c) + \varepsilon]$;

(2) $\min_{t \in [0,1]} \|(u_0, s_0) - \widetilde{g}_0(t)\|_E \leq \sqrt{\varepsilon}$;

(3) $\|\widetilde{J}'_\mu|_{\Sigma_c\times\mathbb{R}}(u_0,s_0)\|_{E^*}\le 2\sqrt{\varepsilon}$, *i.e.*

$$|\langle \widetilde{J}'_\mu(u_0,s_0),z\rangle_{E^*\times E}|\le 2\sqrt{\varepsilon}\,\|z\|_E\,,$$

holds for all $z\in\widetilde{T}_{(u_0,s_0)}:=\{(z_1,z_2)\in E,\langle u_0,z_1\rangle_{L^2}=0\}$.

Proof of Lemma 6.2.6. For each $n\in\mathbb{N}^+$, by the definition of $\gamma_\mu(c)$, there exists a $g_n\in\Gamma_c$ such that

$$\max_{t\in[0,1]}J_\mu(g_n(t))\le\gamma_\mu(c)+\frac{1}{n}.$$

Denote by g_n^* the Schwarz symmetrization of $g_n\in\Gamma_c$. Then by the Polya-Szegö inequality $\|\nabla u^*\|_q^q\le\|\nabla u\|_q^q$, $\forall q\in[1,\infty)$, and using [41, Lemma 4.3], we have that

$$\max_{t\in[0,1]}J_\mu(g_n^*(t))\le\max_{t\in[0,1]}J_\mu(g_n(t)).$$

Since $\widetilde{\gamma}_\mu(c)=\gamma_\mu(c)$, then for each $n\in\mathbb{N}^+$, $\widetilde{g}_n:=(g_n^*,0)\in\widetilde{\Gamma}_c$ and satisfies

$$\max_{t\in[0,1]}\widetilde{J}_\mu(\widetilde{g}_n(t))\le\widetilde{\gamma}_\mu(c)+\frac{1}{n}.$$

Thus applying Lemma 6.2.7, we obtain a sequence $\{(w_n,s_n)\}\subset\Sigma_c\times\mathbb{R}$ such that:

(i) $\widetilde{J}_\mu(w_n,s_n)\in[\gamma_\mu(c)-\frac{1}{n},\gamma_\mu(c)+\frac{1}{n}]$;

(ii) $\min\limits_{t\in[0,1]}\|(w_n,s_n)-(g_n^*(t),0)\|_E\le\frac{1}{\sqrt{n}}$;

(iii) $\|\widetilde{J}'_\mu|_{\Sigma_c\times\mathbb{R}}(w_n,s_n)\|_{E^*}\le\frac{2}{\sqrt{n}}$, i.e.

$$|\langle \widetilde{J}'_\mu(w_n,s_n),z\rangle_{E^*\times E}|\le \frac{2}{\sqrt{n}}\,\|z\|_E\,,$$

holds for all $z\in\widetilde{T}_{(w_n,s_n)}:=\{(z_1,z_2)\in E,\langle w_n,z_1\rangle_{L^2}=0\}$.

Now we claim that for each $n\in\mathbb{N}^+$, there exists a $t_n\in[0,1]$ such that $u_n:=H(w_n,s_n)$ and $v_n:=g_n^*(t_n)$ satisfy (6.2.14). Indeed, first, from (i) we have that $J_\mu(u_n)\to\gamma_\mu(c)$, since $J_\mu(u_n)=J_\mu(H(w_n,s_n))=\widetilde{J}_\mu(w_n,s_n)$. Secondly, by simple calculations, we have

$$Q_\mu(u_n)=\langle\widetilde{J}'_\mu(w_n,s_n),(0,1)\rangle_{E^*\times E},$$

and $(0,1)\in\widetilde{T}_{(w_n,s_n)}$. Thus (iii) yields that $Q_\mu(u_n)\to 0$. To verify that $\|J'_\mu|_{\Sigma_c}(u_n)\|_{X^*}\to 0$, it suffices to prove for $n\in\mathbb{N}^+$ sufficiently large, that

$$|\langle J'_\mu(u_n),\phi\rangle_{X^*\times X}|\le\frac{4}{\sqrt{n}}\,\|\phi\|_X^2\,,\quad\text{for all}\quad \phi\in T_{u_n},\tag{6.2.16}$$

where $T_{u_n}:=\{\phi\in X,\ \langle u_n,\phi\rangle_{L^2}=0\}$. To this end, we note that, for $\phi\in T_{u_n}$, by denoting $\widetilde{\phi}=H(\phi,-s_n)$, one has

$$\begin{aligned}\langle J'_\mu(u_n),\phi\rangle &= \mu\int_{\mathbb{R}^N}|\nabla u_n|^2\nabla u_n\nabla\phi dx+\int_{\mathbb{R}^N}\nabla u_n\nabla\phi dx\\ &+ 2\int_{\mathbb{R}^N}(u_n\phi|\nabla u_n|^2+|u_n|^2\nabla u_n\nabla\phi)dx-\int_{\mathbb{R}^N}|u_n|^{p-1}u_n\phi dx.\\ &= e^{(N+4)s_n}\cdot\mu\int_{\mathbb{R}^N}|\nabla w_n|^2\nabla w_n\nabla\phi dx+e^{2s_n}\int_{\mathbb{R}^N}\nabla w_n\nabla\phi dx\end{aligned}$$

$$
\begin{aligned}
&+ \quad e^{(N+2)s_n} \cdot 2 \int_{\mathbb{R}^N} (w_n\phi|\nabla w_n|^2 + |w_n|^2 \nabla w_n \nabla \phi) dx \\
&- \quad e^{\frac{N(p-1)}{2}s_n} \int_{\mathbb{R}^N} |w_n|^{p-1} w_n \phi dx \;=\; \langle \widetilde{J}'_\mu(w_n, s_n), (\widetilde{\phi}, 0)\rangle_{E^* \times E}.
\end{aligned}
$$

If $(\widetilde{\phi}, 0) \in \widetilde{T}_{(w_n,s_n)}$ and $\|(\widetilde{\phi}, 0)\|_E^2 \le 2\|\phi\|_X^2$ as $n \in \mathbb{N}^+$ sufficiently large, then from (iii) we conclude (6.2.16). To verify this condition, one observes that $(\widetilde{\phi}, 0) \in \widetilde{T}_{(w_n,s_n)} \Leftrightarrow \phi \in T_{u_n}$, also from (ii) it follows that

$$
|s_n| = |s_n - 0| \le \min_{t \in [0,1]} \|(w_n, s_n) - (g_n^*(t), 0)\|_E \le \frac{1}{\sqrt{n}}, \tag{6.2.17}
$$

by which we deduce that

$$
\begin{aligned}
\|(\widetilde{\phi}, 0)\|_E^2 \;&=\; \|\widetilde{\phi}\|_X^2 = \int_{\mathbb{R}^N} |\phi(x)|^2 dx + e^{-2s_n} \int_{\mathbb{R}^N} |\nabla \phi(x)|^2 dx \\
&+ \quad e^{-Ns_n} \int_{\mathbb{R}^N} |\phi(x)|^4 dx + e^{-(N+4)s_n} \int_{\mathbb{R}^N} |\nabla \phi(x)|^4 dx \;\le 2\, \|\phi\|_X^2,
\end{aligned}
$$

holds as $n \in \mathbb{N}^+$ large enough. Thus (6.2.16) has been proved. Finally, we know from (ii) that for each $n \in \mathbb{N}^+$, there exists a $t_n \in [0,1]$, such that $\|(w_n, s_n) - (g_n^*(t_n), 0)\|_E \to 0$. This implies in particular that

$$
\|w_n - g_n^*(t_n)\|_X \to 0.
$$

Thus from (6.2.17) and

$$
\|u_n - v_n\|_X = \|H(w_n, s_n) - g_n^*(t_n)\|_X \le \|H(w_n, s_n) - w_n\|_X + \|w_n - g_n^*(t_n)\|_X,
$$

we conclude that $\|u_n - v_n\|_X \to 0$ as $n \to \infty$. At this point, the proof of the lemma is completed. $\square$

We now derive a similar Palais-Smale sequence for J_μ at the level $\tilde{m}_\mu(c)$. As a first step we prove

Lemma 6.2.8. *For any given $c \in (c_0, c(p, N))$, there exists a minimizing sequence $\{u_n\} \subset \Sigma_c \setminus B$ of Schwarz symmetric functions, such that*

$$
\begin{cases}
J_\mu(u_n) \to \tilde{m}_\mu(c), \\
Q_\mu(u_n) = 0, \;\; \forall\; n \in \mathbb{N}^+.
\end{cases} \tag{6.2.18}
$$

Proof. First we prove that there exists a sequence $\{v_n\} \subset \Sigma_c \setminus B$ satisfying (6.2.18). Let $\{u_n\} \subset \Sigma_c \setminus B$ be such that $J_\mu(u_n) \to \tilde{m}_\mu(c)$, we claim that we may assume that $\{u_n\}$ satisfies $Q_\mu(u_n) = 0$ for all $n \in \mathbb{N}^+$.

Indeed, if $Q_\mu(u_n) = 0$, for some $n \in \mathbb{N}$, we are done. If $Q_\mu(u_n) \neq 0$, then we consider the scaling

$$
u_n^t(x) := t^{N/2} u_n(tx), \;\; \forall\; t > 0. \tag{6.2.19}
$$

Note that for all $t > 0$, $u_n^t \in \Sigma_c$ and direct calculations show that $Q_\mu(u_n^t) = \frac{d}{dt} J_\mu(u_n^t)$ and $Q_\mu(u_n^t) \to \infty$ as $t \to \infty$. Thus if $Q_\mu(u_n) < 0$ there exists by continuity a $t_n^0 > 1$ such that $Q_\mu(u_n^{t_n^0}) = 0$ and

$$
J_\mu(u_n^{t_n^0}) \le J_\mu(u_n). \tag{6.2.20}
$$

Also $u_n^{t_n^0} \in \Sigma_c \setminus B$. If $Q_\mu(u_n) > 0$ we also claim that there exists a $t_n^0 \in (0,1)$, such that $Q_\mu(u_n^{t_n^0}) = 0$ and $J_\mu(u_n^{t_n^0}) \leq J_\mu(u_n)$. To prove the claim first observe that it is not possible to have $Q_\mu(u_n^t) > 0, \ \forall\, t \in (0,1)$ since otherwise there exists a $t_n^* \in (0,1)$ such that $\int_{\mathbb{R}^N}(1+|u_n^{t_n^*}|^2)|\nabla u_n^{t_n^*}|^2 dx = k_0$ and this leads to

$$J_\mu(u_n^{t_n^*}) \leq J_\mu(u_n) \to \tilde{m}_\mu(c) \tag{6.2.21}$$

and

$$J_\mu(u_n^{t_n^*}) \geq \inf_{u\in C_{k_0}} J_\mu(u). \tag{6.2.22}$$

Clearly (6.2.21)-(6.2.22) contradict (6.2.12). We conclude that there exists a $t_n^0 \in (0,1)$ such that

$$Q_\mu(u_n^{t_n^0}) = 0 \ \text{ and } \ J_\mu(u_n^{t_n^0}) \leq J_\mu(u_n).$$

Since by Lemma 6.2.2, $Q_\mu(u) > 0$ for $u \in B$ we also have that $u_n^{t_n^0} \in \Sigma_c \setminus B$. At this point we have shown that it is possible to choose, for each $n \in \mathbb{N}$ a $t_n^0 > 0$ such that $Q_\mu(u_n^{t_n^0}) = 0$, $J_\mu(u_n^{t_n^0}) \to \tilde{m}_\mu(c)$ and $u_n^{t_n^0} \in \Sigma_c \setminus B$.

Now denote by u_n^* the Schwarz symmetrization of u_n and let us prove that $\{u_n^*\} \subset \Sigma_c \setminus B$ and is a minimizing sequence of $\tilde{m}_\mu(c)$. For each $n \in \mathbb{N}$, by the Polya-Szegö inequality $\|\nabla u^*\|_q^q \leq \|\nabla u\|_q^q$, $\forall q \in [1,\infty)$ and also [41, Lemma 4.3] we have

$$J_\mu(u_n^*) \leq J_\mu(u_n), \ \text{ and } \ Q_\mu(u_n^*) \leq Q_\mu(u_n) = 0. \tag{6.2.23}$$

At this point (6.2.3) implies that $u_n^* \in \Sigma_c \setminus B$. This and (6.2.23) lead to

$$\begin{cases} J_\mu(u_n^*) \to \tilde{m}_\mu(c), \\ Q_\mu(u_n^*) \leq 0, \ \forall\, n \in \mathbb{N}^+. \end{cases}$$

If $Q_\mu(u_n^*) < 0$, we may use the above scaling arguments to get a $v_n^* \in \Sigma_c \setminus B$ satisfying (6.2.18). At this point the proof is completed. $\square$

Lemma 6.2.9 (A special Palais-Smale sequence for $\tilde{m}_\mu(c)$)**.** *Assume that $p \in (1+\frac{4}{N}, 3+\frac{4}{N}), N \geq 1$. Then for any given $c \in (c_0, c(p,N))$, for each $\mu \in (0,\mu_0)$ there exists a sequence $\{u_n\} \in \Sigma_c \setminus B$, and a sequence $\{v_n\} \subset \Sigma_c \setminus B$ of Schwarz symmetric functions, such that*

$$\begin{cases} J_\mu(u_n) \to \tilde{m}_\mu(c), \\ \|u_n - v_n\|_X \to 0, \\ \|J'_\mu|_{\Sigma_c}(u_n)\|_{X^*} \to 0, \\ Q_\mu(v_n) = 0, \ \forall\, n \in \mathbb{N}^+, \end{cases} \tag{6.2.24}$$

as $n \to \infty$.

Proof. In Lemma 6.2.8 we have obtained a sequence $\{v_n\} \subset \Sigma_c \setminus B$ of Schwarz symmetric functions, satisfying

$$J_\mu(v_n) \to \tilde{m}_\mu(c) \ \text{ and } Q_\mu(v_n) = 0, \ \forall\, n \in \mathbb{N}^+.$$

It is standard to show that, for any $a > 0$ there exists a $b > 0$ such that

$$J_\mu(u) \geq \inf_{u\in C_{k_0}} J_\mu(u) - a \quad \text{if} \quad u \in \cup_{k\in[k_0-b,k_0+b]} C_k.$$

Thus from (6.2.12) we deduce that $\{v_n\} \subset \Sigma_c \backslash \bigcup_{0<k\leq k_0+b}$ for some $b > 0$ and for $n \in \mathbb{N}$ large enough. Thus, roughly speaking, $\{v_n\}$ stays away from the boundary. At this point we deduce from [53, Corollary 1.3] that there exists a sequence $\{u_n\} \subset \Sigma_c$ such that

$$\begin{cases} J_\mu(u_n) \leq J_\mu(v_n), \\ \|u_n - v_n\|_X \to 0, \\ \|J'_\mu|_{\Sigma_c}(u_n)\|_{X^*} \to 0. \end{cases} \tag{6.2.25}$$

This completes the proof of the Lemma. □

Next we show the compactness of the Palais-Smale sequences obtained in Lemmas 6.2.6 and 6.2.9.

Proposition 6.2.10. *Assume that $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$, $N \geq 1$. Let $\{u_n\} \subset \Sigma_c$ be a Palais-Smale sequence as obtained in Lemmas 6.2.6 or 6.2.9. Then there exist a $u_c \in X\backslash\{0\}$ and a $\lambda_c \in \mathbb{R}$, such that, up to a subsequence,*

1) $u_n \rightharpoonup u_c > 0$, in X;

2) $J'_\mu(u_n) - \lambda_c u_n \to 0$, in X^*;

3) $J'_\mu(u_c) - \lambda_c u_c = 0$, in X^*.

Moreover, if $\lambda_c < 0$, we have that

$$\lim_{n\to\infty} \|u_n - u_c\|_X = 0. \tag{6.2.26}$$

Proof of Proposition 6.2.10. From Lemma 6.2.5 we know that $\{u_n\}$ is bounded in X. This implies in particular the boundedness of the Schwarz symmetric sequences $\{v_n\}$ obtained in Lemmas 6.2.6 or 6.2.9. Thus by [37, Proposition 1.7.1] we conclude that up to a subsequence, there exists a $u_c \in X$, which is nonnegative and Schwarz symmetric, such that

$$v_n \rightharpoonup u_c \geq 0, \quad \text{in } X;$$
$$v_n \to u_c, \quad \text{in } L^q(\mathbb{R}^N),\ \forall\, q \in (2, 2^*).$$

By interpolation, we have that

$$v_n \to u_c, \quad \text{in } L^q(\mathbb{R}^N),\ \forall\, q \in (2, 2\cdot 2^*).$$

In view of $\|u_n - u_c\|_q \leq \|u_n - v_n\|_q + \|v_n - u_c\|_q$, one gets that

$$u_n \to u_c, \quad \text{in } L^q(\mathbb{R}^N),\ \forall\, q \in (2, 2\cdot 2^*). \tag{6.2.27}$$

At this point we shall use the additional information that $Q_\mu(u_n) \to 0$ or $Q_\mu(v_n) = 0$, $\forall\, n \in \mathbb{N}^+$ to show that $u_c \neq 0$. First let $\{u_n\} \subset \Sigma_c$ be the Palais-Smale sequence constructed in Lemma 6.2.6 and assume that $u_c = 0$. Then by (6.2.27) we have that $||u_n||_{p+1} \to 0$ and using that $Q_\mu(u_n) \to 0$ we deduce that

$$\|\nabla u_n\|_4^4 \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} (1+|u_n|^2)|\nabla u_n|^2 dx \to 0.$$

This leads to $J_\mu(u_n) \to 0$, which contradicts the fact that $J_\mu(u_n) \to \gamma_\mu(c) > 0$. Now for the Palais-Smale sequence constructed in Lemma 6.2.9 if we assume that $u_c = 0$ we also get from (6.2.27) and $Q_\mu(v_n) = 0$ that

$$\int_{\mathbb{R}^N} (1+|v_n|^2)|\nabla v_n|^2 dx \to 0.$$

This contradict the fact that $\{v_n\} \subset \Sigma_c \backslash B$. Having proved in both cases that $u_c \neq 0$, Point 1) is established.

Since $\{u_n\} \subset X$ is bounded, following Berestycki and Lions [20, Lemma 3] we know that we know that:

$$\begin{aligned} J'_\mu|_{\Sigma_c}(u_n) &\longrightarrow 0 \text{ in } X^* \\ &\Longleftrightarrow J'_\mu(u_n) - \langle J'_\mu(u_n), u_n \rangle u_n \longrightarrow 0 \text{ in } X^*. \end{aligned}$$

Thus for any $\phi \in X$,

$$\begin{aligned} \langle J'_\mu(u_n) &- \langle J'_\mu(u_n), u_n \rangle u_n, \phi \rangle = \mu \int_{\mathbb{R}^N} |\nabla u_n|^2 \nabla u_n \nabla \phi dx + \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx \\ &+ 2 \int_{\mathbb{R}^N} \Big(u_n \phi |\nabla u_n|^2 + |u_n|^2 \nabla u_n \nabla \phi \Big) dx - \int_{\mathbb{R}^N} |u_n|^{p-1} u_n \phi dx \qquad (6.2.28) \\ &- \lambda_n \int_{\mathbb{R}^N} u_n \phi dx \to 0, \end{aligned}$$

where

$$\lambda_n = \frac{1}{\|u_n\|_2^2} \Big\{ \mu \|\nabla u_n\|_4^4 + \|\nabla u_n\|_2^2 + 4 \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |u_n|^{p+1} dx \Big\}. \qquad (6.2.29)$$

In particular $J'_\mu(u_n)u_n - \lambda_n \|u_n\|_2^2 \to 0$, and it follows that $\{\lambda_n\}$ is bounded since

$$J'_\mu(u_n)u_n = \mu \|\nabla u_n\|_4^4 + \|\nabla u_n\|_2^2 + 4 \int_{\mathbb{R}^N} |u_n|^2 |\nabla u_n|^2 dx - \int_{\mathbb{R}^N} |u_n|^{p+1} dx$$

is bounded. Thus there exists a $\lambda_c \in \mathbb{R}$, such that up to a subsequence, $\lambda_n \to \lambda_c$. This and (6.2.28) imply Point 2). To check Point 3), it is enough, in view of Point 2), to show that for any $\phi \in X$,

$$\langle J'_\mu(u_n) - \lambda_c u_n, \phi \rangle \to \langle J'_\mu(u_c) - \lambda_c u_c, \phi \rangle. \qquad (6.2.30)$$

To prove (6.2.30), note that

$$\begin{aligned} \langle J'_\mu(u_n) - \lambda_c u_n, \phi \rangle &= \mu \int_{\mathbb{R}^N} |\nabla u_n|^2 \nabla u_n \nabla \phi dx + \int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx - \lambda_c \int_{\mathbb{R}^N} u_n \phi dx \\ &+ 2 \int_{\mathbb{R}^N} \Big(u_n \phi |\nabla u_n|^2 + |u_n|^2 \nabla u_n \nabla \phi \Big) dx - \int_{\mathbb{R}^N} |u_n|^{p-1} u_n \phi dx. \end{aligned}$$

Since $u_n \rightharpoonup u_c$ in X, we clearly have

$$\int_{\mathbb{R}^N} \nabla u_n \nabla \phi dx \to \int_{\mathbb{R}^N} \nabla u_c \nabla \phi dx,$$

$$\int_{\mathbb{R}^N} u_n \phi dx \to \int_{\mathbb{R}^N} u_c \phi dx, \ \int_{\mathbb{R}^N} |u_n|^{p-1} u_n \phi dx \to \int_{\mathbb{R}^N} |u_c|^{p-1} u_c \phi dx.$$

Thus we only need to prove that

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 \nabla u_n \nabla \phi dx \to \int_{\mathbb{R}^N} |\nabla u_c|^2 \nabla u_c \nabla \phi dx; \qquad (6.2.31)$$

$$\int_{\mathbb{R}^N} u_n \phi |\nabla u_n|^2 + |u_n|^2 \nabla u_n \nabla \phi dx \to \int_{\mathbb{R}^N} u_c \phi |\nabla u_c|^2 + |u_c|^2 \nabla u_c \nabla \phi dx. \qquad (6.2.32)$$

But $\{|\nabla u_n|^2\nabla u_n\}$ is bounded in $L^{4/3}(\mathbb{R}^N)$ since $\{\nabla u_n\}$ is bounded in $L^4(\mathbb{R}^N)$. Thus $|\nabla u_n|^2\nabla u_n \rightharpoonup |\nabla u|^2\nabla u$ in $L^{4/3}(\mathbb{R}^N)$ and then we get (6.2.31), by weak convergence for any $\nabla\phi \in L^4(\mathbb{R}^N)$. Similarly, using Young inequality, we have that

$$(|u_n||\nabla u_n|^2)^{4/3} \leq \frac{1}{3}|u_n|^4 + \frac{2}{3}|\nabla u_n|^4,$$

$$(|u_n|^2|\nabla u_n|)^{4/3} \leq \frac{2}{3}|u_n|^4 + \frac{1}{3}|\nabla u_n|^4.$$

These yield that both $\{|u_n||\nabla u_n|^2\}$ and $\{|u_n|^2|\nabla u_n|\}$ are bounded in $L^{4/3}(\mathbb{R}^N)$, since $\{u_n\}$ is bounded in X. Thus (6.2.32) holds by a similar argument. At this point, (6.2.30) holds and we have proved Point 3).

Finally, we note from Points 2) and 3) that

$$\langle J'_\mu(u_n) - \lambda_c u_n, u_n\rangle \to \langle J'_\mu(u_c) - \lambda_c u_c, u_c\rangle = 0.$$

Using (6.2.27) we obtain that

$$\begin{aligned}\mu\|\nabla u_n\|_4^4 &+ \|\nabla u_n\|_2^2 - \lambda_c\|u_n\|_2^2 + 4\int |u_n|^2|\nabla u_n|^2dx \to \\ \mu\|\nabla u_c\|_4^4 &+ \|\nabla u_c\|_2^2 - \lambda_c\|u_c\|_2^2 + 4\int |u_c|^2|\nabla u_c|^2dx.\end{aligned}$$

If $\lambda_c < 0$, this implies that $\|u_n - u_c\|_X \to 0$, since $u_n \rightharpoonup u_c$ in X. □

In our next lemma, we discuss the sign of λ_c.

Lemma 6.2.11. *[The sign of λ] Assume that $p \in (1, \frac{N+2}{N-2}]$ if $N \geq 3$ and $p \in (1,\infty)$ if $N = 1,2$. For any $\mu \in [0,\infty)$, if $(u,\lambda) \in X \setminus \{0\} \times \mathbb{R}$ solves the equation*

$$J'_\mu(u) - \lambda u = 0, \tag{6.2.33}$$

then necessarily $\lambda < 0$.

Proof. Let the couple (u,λ) with $u \neq 0$ solves (6.2.33). Then following the proof of [89, Lemma 5.10] one gets that $u \in L^\infty_{loc}(\mathbb{R}^N)$. Thus as in [41, Lemma 3.1], we obtain the following "Pohozaev" identity

$$\frac{\mu(N-4)}{4N}\|\nabla u\|_4^4 + \frac{N-2}{N}\left(\frac{1}{2}\|\nabla u\|_2^2 + \int_{\mathbb{R}^N}|u|^2|\nabla u|^2dx\right) - \frac{\lambda}{2}\|u\|_2^2 = \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \tag{6.2.34}$$

In addition, testing (6.2.33) by u, we have

$$\mu\|\nabla u\|_4^4 + ||\nabla u||_2^2 + 4\int_{\mathbb{R}^N}|u|^2|\nabla u|^2dx - \lambda||u||_2^2 - ||u||_{p+1}^{p+1} = 0. \tag{6.2.35}$$

Thus by a simple calculation, it follows from (6.2.34) and (6.2.35) that

$$\begin{aligned}\lambda\|u\|_2^2 &= \mu\cdot\frac{(N-4)p-(3N+4)}{2N(p-1)}\|\nabla u\|_4^4 \\ &+ \frac{(N-2)p-(N+2)}{N(p-1)}\|\nabla u\|_2^2 \\ &+ 2\frac{(N-2)p-(3N+2)}{N(p-1)}\int_{\mathbb{R}^N}|u|^2|\nabla u|^2dx.\end{aligned} \tag{6.2.36}$$

This proves that $\lambda < 0$ if $p > 0$ satisfies the assumption of the lemma. □

Based on the above preliminary works, we conclude that

Theorem 6.2.12. *Assume that $p \in (1 + \frac{4}{N}, 3 + \frac{4}{N})$ if $N = 1, 2, 3$ and $p \in (1 + \frac{4}{N}, \frac{N+2}{N-2}]$ if $N \geq 4$. Then there exists a $c_0 \in (0, c(p, N))$ such that*

(1) For any $c \in (c_0, \infty)$ there exists a $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$, the functional J_μ has a critical point u_c on Σ_c at the level $\gamma_\mu(c)$.

(2) For any $c \in (c_0, c(p, N))$ there exists a $\mu_0 > 0$ such that for each $\mu \in (0, \mu_0)$, the functional J_μ has a critical point v_c on Σ_c at the level $\tilde{m}_\mu(c)$.

In addition, u_c and v_c are Schwarz symmetric and there exist a $\lambda_c < 0$ and a $\beta_c < 0$ such that

$$J'_\mu(u_c) - \lambda_c u_c = 0 \quad \text{and} \quad J'_\mu(v_c) - \beta_c v_c = 0.$$

Proof of Theorem 6.2.12. Combining Proposition 6.2.10 and Lemma 6.2.11, the theorem follows. □

6.3 Convergence issues

In this section, letting $\mu \to 0$, we show that the sequences of critical points of J_μ obtained in Theorem 6.2.12 converge to a critical point of $J = J_0$ on $\bar{S}(c)$.

Theorem 6.3.1. *[Convergence issues] Assume that $\mu_n \underset{n\to\infty}{\to} 0$. For some $c > 0$, let $\{w_n\} \subset \Sigma_c$ be a sequence of Schwarz symmetric functions, and $\{\lambda_n\} \subset \mathbb{R}^-$, satisfying that*

$$|J_{\mu_n}(w_n)| \leq C, \quad \text{and} \quad J'_{\mu_n}(w_n) - \lambda_n w_n = 0,$$

where $C > 0$ is independent of $n \in \mathbb{N}$. Then there exist a $w_c \in W^{1,2} \cap L^\infty(\mathbb{R}^N)$ and a $\lambda_c \in \mathbb{R}$, such that up to a subsequence, as $n \to \infty$, we have that

$$\begin{aligned} &\lambda_n \to \lambda_c, \ \text{in } \mathbb{R}, \\ &J'(w_c) - \lambda_c w_c = 0. \end{aligned} \tag{6.3.1}$$

Moreover, if $\lambda_c < 0$, then

$$\begin{aligned} w_n &\to w_c, \quad \text{in } W^{1,2}(\mathbb{R}^N), \\ w_n \nabla w_n &\to w_c \nabla w_c, \quad \text{in } L^2(\mathbb{R}^N), \\ \mu_n \|\nabla w_n\|_4^4 &\to 0, \end{aligned} \tag{6.3.2}$$

as $n \to \infty$. Thus $w_c \in W^{1,2} \cap L^\infty(\mathbb{R}^N)$ is a critical point of J on $\bar{S}(c)$.

Proof. To show this theorem, we borrow ideas from the proof of [86, Theorem 1.1]. First, since $|J_{\mu_n}(w_n)| \leq C$ and $J'_{\mu_n}(w_n) - \lambda_n w_n = 0$, we observe from the proofs of Lemma 6.2.5 and Proposition 6.2.10 that $\{\int_{\mathbb{R}^N} (1 + |w_n|^2)|\nabla w_n|^2 dx\}$ is bounded and $\{\lambda_n\}$ is bounded. Thus up to a subsequence, $\lambda_n \to \lambda_c \in \mathbb{R}$, and noting that $\{w_n\} \subset \Sigma_c$ is Schwarz symmetric, by [37, Proposition 1.7.1] we obtain, up to a subsequence that

$$\begin{aligned} w_n &\rightharpoonup w_c, \ \text{in } W^{1,2}(\mathbb{R}^N), \\ w_n &\to w_c, \ \text{in } L^q(\mathbb{R}^N), \ \forall q \in (2, 2 \cdot 2^*), \\ w_n \nabla w_n &\rightharpoonup w_c \nabla w_c, \ \text{in } L^2(\mathbb{R}^N), \\ w_n &\to w_c, \ \text{a.e. in } \mathbb{R}^N, \end{aligned} \tag{6.3.3}$$

for some $w_c \in W^{1,4} \cap W^{1,2}(\mathbb{R}^N)$. Since $\{w_n\}$ satisfies $J'_{\mu_n}(w_n) - \lambda_n w_n = 0$, thus we have

$$\begin{aligned}
&\mu_n \int_{\mathbb{R}^N} |\nabla w_n|^2 \nabla w_n \nabla \phi dx + \int_{\mathbb{R}^N} \nabla w_n \nabla \phi dx - \lambda_n \int_{\mathbb{R}^N} w_n \phi dx \\
&+2 \int_{\mathbb{R}^N} \Big(w_n \phi |\nabla w_n|^2 + |w_n|^2 \nabla w_n \nabla \phi \Big) dx = \int_{\mathbb{R}^N} |w_n|^{p-1} w_n \phi dx,
\end{aligned} \tag{6.3.4}$$

for any $\phi \in W^{1,4} \cap W^{1,2}(\mathbb{R}^N)$. Then by the Sobolev inequality and Moser iteration we may get

$$\|w_n\|_{L^\infty(\mathbb{R}^N)} \le C, \quad \text{and } \|w_c\|_{L^\infty(\mathbb{R}^N)} \le C. \tag{6.3.5}$$

We now show that w_c satisfies that

$$\langle J'(w_c) - \lambda_c w_c, \phi \rangle = 0, \ \forall \ \phi \in W^{1,2} \cap L^\infty(\mathbb{R}^N). \tag{6.3.6}$$

In (6.3.4) we choose $\phi = \psi \exp(-w_n)$ with $\psi \in C_0^\infty(\mathbb{R}^N)$, $\psi \ge 0$. Then we have that

$$\begin{aligned}
0 &= \mu_n \int_{\mathbb{R}^N} |\nabla w_n|^2 \nabla w_n (\nabla \psi \exp(-w_n) - \psi \exp(-w_n) \nabla w_n) dx \\
&+ \int_{\mathbb{R}^N} \nabla w_n (\nabla \psi \exp(-w_n) - \psi \exp(-w_n) \nabla w_n) dx \\
&+ 2 \int_{\mathbb{R}^N} |w_n|^2 \nabla w_n (\nabla \psi \exp(-w_n) - \psi \exp(-w_n) \nabla w_n) dx \\
&+ 2 \int_{\mathbb{R}^N} w_n \psi \exp(-w_n) |\nabla w_n|^2 dx - \lambda_n \int_{\mathbb{R}^N} u_n \psi \exp(-w_n) dx \\
&- \int_{\mathbb{R}^N} |w_n|^{p-1} w_n \psi \exp(-w_n) dx.
\end{aligned}$$

This implies that

$$\begin{aligned}
0 &\le \mu_n \int_{\mathbb{R}^N} |\nabla w_n|^2 \nabla w_n \nabla \psi \exp(-w_n) dx + \int_{\mathbb{R}^N} (1 + 2w_n^2) \nabla \psi \nabla w_n \exp(-w_n) dx \\
&- \int_{\mathbb{R}^N} (1 + 2w_n^2 - 2w_n) \psi \exp(-w_n) |\nabla w_n|^2 dx \\
&- \lambda_n \int_{\mathbb{R}^N} w_n \psi \exp(-w_n) dx - \int_{\mathbb{R}^N} |w_n|^{p-1} w_n \psi \exp(-w_n) dx.
\end{aligned}$$

By using (6.3.3) and the fact that $\{\mu_n \|\nabla w_n\|_4^4\}$ is bounded, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla w_c \nabla \left(\psi \exp(-w_c)\right) dx + \int_{\mathbb{R}^N} 2|w_c|^2 \nabla w_c \nabla \left(\psi \exp(-w_c)\right) dx \\
&+ \int_{\mathbb{R}^N} 2w_c \left(\psi \exp(-w_c)\right) |\nabla w_c|^2 dx - \lambda_c \int_{\mathbb{R}^N} w_c \left(\psi \exp(-w_c)\right) dx \\
&\qquad \ge \int_{\mathbb{R}^N} |w_c|^{p-1} w_c \left(\psi \exp(-w_c)\right) dx,
\end{aligned} \tag{6.3.7}$$

in which we also used Fatou's lemma, to get

$$\liminf_n \int_{\mathbb{R}^N} (1 + 2w_n^2 - 2w_n) |\nabla w_n| \psi \exp(-w_n) dx \ge \int_{\mathbb{R}^N} (1 + 2w_c^2 - 2w_c) |\nabla w_c| \psi \exp(-w_c) dx.$$

Let $\chi \ge 0, \chi \in C_0^\infty(\mathbb{R}^N)$. Choose a sequence of nonnegative functions $\psi_n \in C_0^\infty(\mathbb{R}^N)$ such that $\psi_n \to \chi \exp(w_c)$ in $W^{1,2}(\mathbb{R}^N)$, $\psi_n \to \chi \exp(w_c)$ a.e. in $\mathbb{R}^N$, and ψ_n is uniformly bounded in $L^\infty(\mathbb{R}^N)$. Then we get from (6.3.7) that

$$\int_{\mathbb{R}^N} \nabla w_c \nabla \chi dx + 2 \int_{\mathbb{R}^N} \Big(|w_c|^2 \nabla w_c \nabla \chi + w_c \chi |\nabla w_c|^2 \Big) dx - \lambda_c \int_{\mathbb{R}^N} w_c \chi dx \ge \int_{\mathbb{R}^N} |w_c|^{p-1} w_c \chi dx.$$

Similarly by choosing $\phi = \psi \exp(w_n)$, we get an opposite inequality. Thus we obtain that for any $\chi \in C_0^\infty(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \nabla w_c \nabla \chi dx + 2 \int_{\mathbb{R}^N} \Big(|w_c|^2 \nabla w_c \nabla \chi + w_c \chi |\nabla w_c|^2 \Big) dx \\ - \lambda_c \int_{\mathbb{R}^N} w_c \chi dx = \int_{\mathbb{R}^N} |w_c|^{p-1} w_c \chi dx. \tag{6.3.8}$$

This proves (6.3.1).

Now by approximation again, we get from (6.3.8) that

$$\int_{\mathbb{R}^N} |\nabla w_c|^2 dx + 4 \int_{\mathbb{R}^N} |w_c|^2 |\nabla w_c|^2 dx - \lambda_c \int_{\mathbb{R}^N} |w_c|^2 dx = \int_{\mathbb{R}^N} |w_c|^{p+1} dx. \tag{6.3.9}$$

In (6.3.4), we use $\phi = w_n$ to get that

$$\mu_n \int_{\mathbb{R}^N} |\nabla w_n|^4 dx + \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + 4 \int_{\mathbb{R}^N} |w_n|^2 |\nabla w_n|^2 dx \\ - \lambda_n \int_{\mathbb{R}^N} |w_n|^2 dx = \int_{\mathbb{R}^N} |w_n|^{p+1} dx. \tag{6.3.10}$$

and then

$$\mu_n \int_{\mathbb{R}^N} |\nabla w_n|^4 dx + \int_{\mathbb{R}^N} |\nabla w_n|^2 dx + 4 \int_{\mathbb{R}^N} |w_n|^2 |\nabla w_n|^2 dx \\ - \lambda_c \int_{\mathbb{R}^N} |w_n|^2 dx = \int_{\mathbb{R}^N} |w_n|^{p+1} dx + o(1), \tag{6.3.11}$$

since $\lambda_n \to \lambda_c$ and $\int_{\mathbb{R}^N} |w_n|^2 dx = c > 0$. Hence, if $\lambda_c < 0$, using $\int_{\mathbb{R}^N} |w_n|^{p+1} dx \to \int_{\mathbb{R}^N} |w_c|^{p+1} dx$ in (6.3.3), we conclude from (6.3.3) (6.3.9) and (6.3.11) that, as $n \to \infty$,

$$\mu_n \int_{\mathbb{R}^N} |\nabla w_n|^4 dx \to 0, \ \int_{\mathbb{R}^N} |\nabla w_n|^2 dx \to \int_{\mathbb{R}^N} |\nabla w_c|^2 dx,$$

$$\int_{\mathbb{R}^N} |w_n|^2 |\nabla w_n|^2 dx \to \int_{\mathbb{R}^N} |w_c|^2 |\nabla w_c|^2 dx, \ \int_{\mathbb{R}^N} |w_n|^2 dx \to \int_{\mathbb{R}^N} |w_c|^2 dx.$$

Thus from (6.3.1) and (6.3.5) we deduce that $w_c \in W^{1,2} \cap L^\infty(\mathbb{R}^N)$ is a critical point of J on $\bar{S}(c)$. At this point, the proof is completed. □

Now we are able to end the proof of Theorem 6.1.2.

Proof of Theorem 6.1.2. In the case $c \in [c(p,N), \infty)$ the critical point v_c is just the global minimizer already obtained in [41, 65] whose existence is recalled in Lemma 6.1.1. Thus to prove Theorem 6.1.2, thanks to Theorems 6.2.12, 6.3.1 and Lemma 6.2.11 one only needs to show that there exists a $C > 0$ independent of $\mu > 0$, such that

$$|J_\mu(u_c)| \leq C \quad \text{and} \quad |J_\mu(v_c)| \leq C \tag{6.3.12}$$

where u_c and v_c are obtained in Theorem 6.2.12. To prove (6.3.12), note that by definition of $\gamma_\mu(c)$ we have $0 < J_\mu(u_c) = \gamma_\mu(c) \leq \gamma_1(c)$, in which $\gamma_1(c)$ is independent of $\mu > 0$. Also when $c \in (c_0, c(p,N))$ we have $0 \leq J_\mu(v_c) \leq J_\mu(u_c) \leq \gamma_1(c)$. At this point, we have proved the theorem. □

Proof of Lemma 6.1.4. Fix a $c_0 > 0$ large and let $v_0 \in \bar{S}(c_0)$ be fixed. We consider for $t > 0$ the scaling $v_0^t(x) := t^\alpha v_0(t^\beta x)$, where

$$\alpha = \frac{1}{3N+4-Np}, \quad \beta = \frac{p-(3+\frac{2}{N})}{3N+4-Np}.$$

Then

$$\|v_0^t\|_2^2 = t\|v_0\|_2^2, \quad \|\nabla v_0^t\|_2^2 = t^{\lambda_1+1}\|\nabla v_0\|_2^2,$$
$$\int_{\mathbb{R}^N} |v_0^t|^2|\nabla v_0^t|^2 dx = t^{\lambda_2+1}\int_{\mathbb{R}^N} |v_0|^2|\nabla v_0|^2 dx,$$
$$\int_{\mathbb{R}^N} |v_0^t|^{p+1} dx = t^{\lambda_3+1}\int_{\mathbb{R}^N} |v_0|^{p+1} dx,$$

where

$$\lambda_1 = \frac{2p-6-\frac{4}{N}}{3N+4-Np}, \ \lambda_2 = \frac{2p-4-\frac{4}{N}}{3N+4-Np}, \ \lambda_3 = \frac{p-1}{3N+4-Np}.$$

We observe that $\lambda_3 > 0$ and $\lambda_3 > \max\{\lambda_1, \lambda_2\}$ if $p \in (1, 3+\frac{4}{N})$. Also $v_0^t \in \bar{S}(tc_0)$ for all $t > 0$, and

$$\frac{J(v_0^t)}{tc_0} = \frac{1}{c_0}\cdot\left(\frac{t^{\lambda_1}}{2}\|\nabla v_0\|_2^2 + t^{\lambda_2}\int_{\mathbb{R}^N}|v_0|^2|\nabla v_0|^2 dx - \frac{t^{\lambda_3}}{p+1}\int_{\mathbb{R}^N}|v_0|^{p+1}dx\right).$$

This implies that

$$\limsup_{t\to\infty}\frac{m(tc_0)}{tc_0} \le \limsup_{t\to\infty}\frac{J(v_0^t)}{tc_0} = -\infty,$$

from which we deduce that if $v_c \in \bar{S}(c)$ is a global minimizer of J on $\bar{S}(c)$ then $\dfrac{J(v_c)}{c} \to -\infty$ as $c \to \infty$. At this point recalling, see the proof of [41, Lemma 4.6], that

$$J(v_c) = \frac{1}{N}\left(\|\nabla v_c\|_2^2 + \int_{\mathbb{R}^N}|v_c|^2|\nabla v_c|^2 dx\right) + \frac{\beta_c}{2}\|v_c\|_2^2$$

we deduce that $\beta_c \to -\infty$ as $c \to \infty$ uniformly. At this point, the lemma is proved. □

6.4 Relationship between ground states and global minimizers on the constraint

In this section, we prove Theorem 6.1.5 which gives a relationship between the ground states of (P_λ) and the global minimizers of $\bar{m}(c)$.

We recall from Lemma 6.1.1 and [41, Lemma 4.6] that when (p, c, N) satisfies the following conditions:

(i) $c \in (0, \infty)$, and $p \in (1, 1+\frac{4}{N}), N \ge 1$,

(ii) $c \in [c(p,N), \infty)$, and $p \in (1+\frac{4}{N}, 3+\frac{4}{N}), N \ge 1$,

there exist a global minimizer v_c of $\bar{m}(c)$ and a Lagrange multiplier $\beta_c < 0$, such that (v_c, β_c) is a solution of (P_λ). Also we know from [41, Theorem 1.3] that for $\lambda = \beta_c < 0$, the equation (P_λ) has a ground state solution. We denote

$$\mathcal{A}_\lambda := \Big\{u : \ u \text{ is a solution of } (P_\lambda)\Big\},$$

$$\mathcal{G}_\lambda := \Big\{u : \ u \text{ is a ground state solution of } (P_\lambda)\Big\}.$$

Proof of Theorem 6.1.5. For $\lambda = \beta_c < 0$, let φ_{β_c} be a ground state of (P_λ). Namely, φ_{β_c} solves the minimization problem

$$l_{\beta_c} := \inf\{I_{\beta_c}(u) : \; u \in \mathcal{A}_{\beta_c}\},$$

Since $v_c \in \mathcal{A}_{\beta_c}$, one only needs to show that

$$I_{\beta_c}(v_c) = l_{\beta_c}. \tag{6.4.1}$$

By definition of l_{β_c}, to check (6.4.1) it is enough to show that $I_{\beta_c}(v_c) \leq l_{\beta_c}$. In turn this holds if one can find a $\psi \in \bar{S}(c)$ such that

$$I_{\beta_c}(\psi) \leq I_{\beta_c}(\varphi_{\beta_c}), \tag{6.4.2}$$

since $I_{\beta_c}(v_c) \leq I_{\beta_c}(\psi) \leq I_{\beta_c}(\varphi_{\beta_c}) = l_{\beta_c}$.

To choose $\psi \in \bar{S}(c)$ satisfying (6.4.2), we consider the scaling $u_t(x) := \varphi_{\beta_c}(x/t), t > 0$. Then $\|u_t\|_2^2 = t^N \|\varphi_{\beta_c}\|_2^2$ and by the identities (6.2.34) and (6.2.35), we have

$$I_{\beta_c}(u_t) = \Big(t^{N-2} - \frac{N-2}{N} t^N\Big) \cdot \Big[\frac{1}{2}\|\nabla \varphi_{\beta_c}\|_2^2 + \int_{\mathbb{R}^N} |\varphi_{\beta_c}|^2 |\nabla \varphi_{\beta_c}|^2 dx\Big].$$

Thus

$$\frac{d}{dt} I_{\beta_c}(u_t) = (N-2)(1-t^2)t^{N-3} \left[\frac{1}{2}\|\nabla \varphi_{\beta_c}\|_2^2 + \int_{\mathbb{R}^N} |\varphi_{\beta_c}|^2 |\nabla \varphi_{\beta_c}|^2 dx\right].$$

This implies that when $N \geq 2$

$$I_{\beta_c}(u_t) \leq I_{\beta_c}(\varphi_{\beta_c}), \;\; \forall t > 0.$$

Choosing a suitable $t_0 > 0$ such that $u_{t_0} \in \bar{S}(c)$ and letting $\psi = u_{t_0}$ we obtain (6.4.2). When $N = 1$, since by [41, Theorem 1.3] the non negative solutions of (P_λ) for fixed $\lambda > 0$ are unique the conclusion holds automatically. This completes the proof. $\square$

Chapter 7

Some remarks and perspectives

7.1 Remarks

As we already mentioned, this thesis is mainly devoted to the search of constrained critical points which are not global minimizers of the associated functional. For instance mountain pass type solutions in Theorems 1.1.6 or 1.2.4, or local minimizers also in Theorem 1.2.4. As we have seen throughout the thesis to prove that such critical points exist we face difficulties which are not present when one searches for a global minimizer of the functional. In particular the fact that our suspected critical levels are strictly positive makes delicate the application of the concentration compactness principle of P. L. Lions [83], at least in its classical forms. In trying to show that the weak limits of our Palais-Smale sequences remain on the constraint, we have encountered, for the equations (E_λ) and (P_λ), the need to show that the associated Lagrange parameters are strictly negative. A link between this property and the fact that $c \to \gamma(c)$ is strictly decreasing (a condition which guarantees there the strong convergence) was pointed out in Chapter 3. Indications are also given there that when the Lagrange multiplier is not strictly negative the critical point may not exist. However much remains to be understood on the relationship between the sign of Lagrange multipliers and the fact that critical points are reached or not.

7.2 Perspectives

We end this thesis by mentioning some open problems directly related to the content of the thesis.

7.2.1 On the Schrödinger-Poisson-Slater equations (E_λ)

Concerning the Schrödinger-Poisson-Slater equations

$$-\Delta u - \lambda u + (|x|^{-1} * |u|^2)u - |u|^{p-2}u = 0, \quad \text{in } \mathbb{R}^3. \qquad (E_\lambda)$$

where $p \in (2, 6)$, we believe the following questions are worth of interest.

(1) About the minimization problem of

$$m(c) := \inf_{u \in S(c)} F(u),$$

(see (1.1.7) for the precise definition), as we pointed out in Remark 2.1.5, when $p \in (2, 3)$, it is open whether a minimizer of $m(c)$ exists or not for $c > 0$ large.

In trying to develop a minimization process one faces the difficulty to remove the possible dichotomy of the minimizing sequences. Our hope is that the techniques developed in this thesis can be useful on that problem.

(2) From Theorem 2.1.2 we know that when $p \in (3, \frac{10}{3})$, for each $c \in (c_1, \infty)$, $m(c) < 0$ and admits a minimizer $u_c \in S(c)$ which is a critical point of F on $S(c)$. Using the estimates of Lemma 2.2.2, one can prove that, for each $c \in (c_1, \infty)$, the energy functional F has a mountain pass geometry on the constraint $S(c)$, namely for any given $c > 0$, there exists a point $u_0 \in S(c)$ such that

$$\gamma(c) := \inf_{g \in \Gamma_c} \max_{t \in [0,1]} F(g(t)) > \max\{F(g(0)), F(g(1))\},$$

holds in the set

$$\Gamma_c := \Big\{ g \in C([0,1], S(c)) : \ g(0) = u_0, \ g(1) = u_c \Big\}.$$

Since the functional F is coercive on $S(c)$ (see [14, Lemma 3.1]), any Palais-Smale sequence for F at the level $\gamma(c)$ is bounded. Actually from the variational point of view all these look similarly to when happen for equation (P_λ) when $p \in (1+\frac{4}{N}, 3+\frac{4}{N})$ and one could hope to find a second critical point for F on $S(c)$ of Mountain-Pass type. However we are not enable so far to show that the suspected associated Lagrange multiplier is strictly negative. Namely for this problem we fail to prove the analogue of Lemma 6.2.11.

(3) Let us mention, as possible extension of this thesis, the study of the following stationary fractional Schrödinger equation with a Coulombic potential

$$(-\Delta)^{1/2} u + \lambda u - \kappa(|x|^{-1} * |u|^2) u = 0, \quad \text{in } \mathbb{R}^3 \tag{7.2.1}$$

where $\lambda \in \mathbb{R}$ and $\kappa > 0$. We refer to the literature [39, 82, 111] and the references given there, for the studies of (7.2.1) or of some extended versions. In the cited references, solutions of (7.2.1) are obtained as minimizers of the following minimization problem

$$M_c := \inf \Big\{ \| |\xi|^{1/2} \mathcal{F}[u](\xi) \|_2^2 - \kappa \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 |u(y)|^2}{|x-y|} dx dy, \tag{7.2.2}$$
$$: \ u \in H^{1/2}(\mathbb{R}^3), \int_{\mathbb{R}^3} |u|^2 = c, c > 0 \Big\},$$

where $\mathcal{F}$ denotes the Fourier transform.

It is proved that for any given $c > 0$, M_c admits a minimizer $u_c \in H^{1/2}(\mathbb{R}^3)$ with $\|u_c\|_2^2 = c$. Thus there exists a Lagrange multiplier $\lambda_c \in \mathbb{R}$, such that (u_c, λ_c) solves (7.2.1). Also the orbital stability of the standing waves $e^{-i\lambda t} u_c(x)$ of the Cauchy problem associated to (7.2.1), is established. Equation (7.2.1) differs from (E_λ) in the sense that the replacement of $-\Delta$ by $-(\Delta)^{1/2}$ brings new technicalities. However from the variational point of view, because of the requirement that $\kappa > 0$, the functional associated to (7.2.1) is more simple to handle. It is an open question if the type of results, in particular the existence of saddle point type solutions, obtained in Chapters 2 to 4 could be also derived for (7.2.1) when $\kappa < 0$.

7.2.2 Quasi-linear Schrödinger equations (P_λ)

About the quasi-linear Schrödinger equations

$$-\Delta u - u\Delta(u^2) - \lambda u - |u|^{p-1}u = 0, \quad \text{in } \mathbb{R}^N, \tag{P_λ}$$

the following questions are directly related to Chapter 6.

(1) The restriction $p \leq \frac{N+2}{N-2}$ when $N \geq 4$ in Theorem 6.1.2 only comes from the fact that we need the Lagrange multiplier to be strictly negative (see Lemma 6.2.11). Can this range be extended or a proof Theorem 6.1.2 which do not use the sign of the Lagrange multiplier be given ?

(2) In Theorem 6.1.2 we prove that there exists an interval $c \in (c_0, c(p,N))$ in which we have a mountain pass geometry. Clearly this result is not optimal. In particular it would be interesting to know if the set $c \in (0, c(p,N))$ for which we can find two critical points is an interval (see Remark 6.1.3 in that direction).

(3) In Lemma 6.1.4 we prove an asymptotic result for the Lagrange multiplier associated to minimizers. We conjecture that for the Lagrange multipliers λ_c associated to our mountain pass solution one has $\lambda_c \to 0$ as $c \to \infty$.

(4) We manage to show on problem (P_λ) that there exists two solutions having the same L^2-norm. What are the key ingredients which lead to this result ? What kind of other problems can we hope to treat with our approach ?

As a more remote aim we would like to address the question of the orbital stability of the two solutions we have obtained in Theorem 6.1.2. Note first that concerning the orbital stability or instability of the standing waves associated to the ground states of (P_λ) the situation is still in a developing state. Up to our knowledge, it is only known that when $p \in (3 + \frac{4}{N}, \frac{3N+2}{N-2})$, all ground states of (P_λ) lead to standing waves which are orbitally unstable by blowup in a suitably regular Sobolev space (see [38, 41]).

For simplicity we can first assume that $N = 1$. In this dimension our approximation procedure is not necessary since the functional J is already C^1. Also we know that the ground state solutions are unique when $N = 1$, see [41, Theorem 1.3]. We conjecture, see the drawing below, that the mountain pass solution is orbitally unstable and the local minimizer is orbitally stable. Of course we already know that the global minimizers are orbitally stable, see [41]. Note that to prove that the mountain pass solution is unstable it seems not possible to use the classical approach of H. Berestycki and T. Cazenave [18] applied in Theorem 1.1.16. The fact that constrained critical points exist at an energy level less than our mountain pass value seems a major obstacle. Perhaps to handle these questions of stability other techniques, not variational, will be necessary.

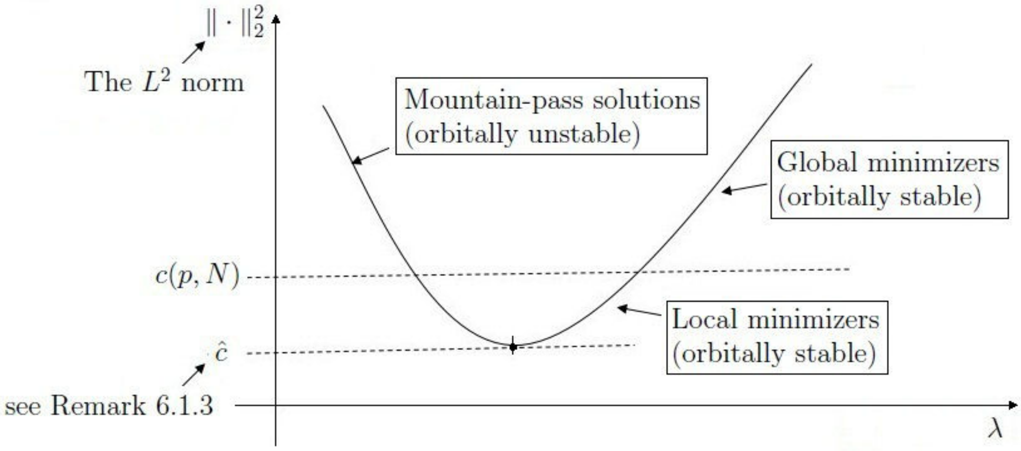
$\|\cdot\|_2^2$
The L^2 norm
Mountain-pass solutions
(orbitally unstable)
Global minimizers
(orbitally stable)
$c(p,N)$
Local minimizers
(orbitally stable)
$\hat{c}$
see Remark 6.1.3
λ

Figure 7.1

Bibliographie

[1] S. Adachi, T. Watanabe, *Uniqueness of ground state solutions for quasilinear Schrödinger equations*, Nonlinear Analysis : T. M. A. 75 (2012), no. 2, 819-833.

[2] R. A. Adams, *Sobolev spaces*, Academic Press, New York-London, 1975.

[3] A. Ambrosetti, A. Malchiodi, D. Ruiz, *Bound states of nonlinear Schrödinger equations with potential vanishing at infinity*, J. Anal. Math. 98 (2006), no. 1, 317-348.

[4] A. Ambrosetti, D. Ruiz, *Multiple bound states for the Schrödinger-Poisson problem*, Commun. Contemp. Math. 10 (2008), no. 3, 391-404.

[5] A. Ambrosetti, Z.-Q. Wang, *Positive solutions to a class of quasilinear elliptic equations on* $\mathbb{R}$, Disc. Cont. Dyna. Syst. 9 (2003), 55-68.

[6] J. Asch and A. Joye, *Mathematical Physics of Quantum Mechanics*, Springer, 2006.

[7] A. Azzollini, P. d'Avenia, A. Pomponio, *On the Schrödinger-Maxwell equations under the effect of a general nonlinear term*, Ann. Inst. H. Poincaré Anal. Non. Linéaire 27 (2010), no. 2, 779-791.

[8] A. Azzollini, P. d'Avenia, A. Pomponio, *Multiple critical points for a class of nonlinear functionals*, Ann. Mat. Pura Appl. (4) 190 (2011), no. 3, 507-523.

[9] A. Azzollini, A. Pomponio, *Ground state solutions for the nonlinear Schrödinger-Maxwell equations*, J. Math. Anal. Appl. 345 (2008), no. 1, 90-108.

[10] C. Bardos, F. Golse, A. D. Gottlieb, N. Mauser, *Mean field dynamics of fermions and the time-dependent Hartree-Fock equation*, J. Math. Pures Appl. (9) 82 (2003), no. 6, 665-683.

[11] T. Bartsch, *Infnitely many solutions of a symmetric Dirichlet Problem*, Nonlin. Anal. 20 (1993), 1205-1216.

[12] T. Bartsch, S. De Valeriola, *Normalized solutions of nonlinear Schrödinger equations*, Arch. Math. 100 (2013), 75-83.

[13] F. G. Bass, N. N. Nasanov, *Nonlinear electromagnetic spin waves*, Physics Reports 189 (1990), 165-223.

[14] J. Bellazzini, G. Siciliano, *Stable standing waves for a class of nonlinear Schrödinger-Poisson equations*, Z. Angew. Math. Phys. 62 (2011), no. 2, 267-280.

[15] J. Bellazzini, G. Siciliano, *Scaling properties of functionals and existence of constrained minimizers*, J. Funct. Anal. 261 (2011), 2486-2507.

[16] J. Bellazzini, L. Jeanjean, T-J. Luo, *Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations*, Proc. London Math. Soc. (3) 107 (2013) 303-339.

[17] V. Benci, D. Fortunato, *An eigenvalue problem for the Schrödinger-Maxwell equations*, Top. Meth. Nonlinear Anal. 11 (1998), 283-293.

[18] H. Berestycki, T. Cazenave, *Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaire*, C. R. Acad. Sci. Paris Sér. I Math. 293 (1981), no. 9, 489-492.

[19] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations I, Existence of a ground state*, Arch. Ration. Mech. Anal. 82, (1983), no. 4, 313-346.

[20] H. Berestycki, P. L. Lions, *Nonlinear scalar field equations II, Existence of infinitely many solutions*, Arch. Rat. Mech. Anal. 82 (1983), no. 4, 347-375.

[21] O. Bokanowski, J. L. López and J. Soler, *On an exchange interaction model for quantum transport : the Schrödinger-Poisson-System*, Math. Models. Methods. Appl. Sci. 8 (2003), 1185-1217.

[22] A. V. Borovskii, A. L. Galkin, *Dynamical modulation of an ultrashort high-intensity laser pulse in matter*, JETP 77 (1993), 562-573.

[23] H. Brezis, *Analyse fonctionnelle, Théorie et applications*, Collection Mathématiques Appliquées pour la Maîtrise, Masson, Paris, 1983.

[24] H. Brezis, E. Lieb, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), no. 3, 486-490.

[25] H. Brezis, A. C. Ponce, *Kato's inequality when Δu is a measure (English, French summary)*, C. R. Math. Acad. Sci. Paris 338 (2004), no. 8, 599-604.

[26] Y. Brihaye and B. Hartmann, *Solitons on nanotubes and fullerenes as solutions of a modified nonlinear Schrödinger equation*, Advances in Soliton Research 2006 (Hauppauge, NY : Nova Sci.Publ.) 135-151.

[27] Y. Brihaye, B. Hartmann and W. J. Zakrzewski, *Spinning solitons of a modified nonlinear Schrödinger equation*, Phys. Rev. D 69 (2004), 087701.

[28] L. Brizhik, A. Eremko, B. Piette and W. J. Zakrzewski, *Electron self-trapping in a discrete two-dimensional lattice*, Physica D 159 (2001), 71-90.

[29] L. Brizhik, A. Eremko, B. Piette and W. J. Zakrzewski, *Static solutions of a D-dimensional modified nonlinear Schrödinger equation*, Nonlinearity 16 (2003), 1481-1497.

[30] J. Byeon, L. Jeanjean, M. Mariş, *Symmetry and monotonicity of least energy solutions*, Calc. Var. Partial Differential Equations 36 (2009), no. 4, 481-492.

[31] P. E. Conner and E. E. Floyd, *Fixed point free involutions and equivariant maps. II.*, Trans. Amer. Math. Soc. 105, 222-228.

[32] M. Caliari, M. Squassina, *On a bifurcation value related to quasilinear Schrödinger equations*, J. Fixed Point Theory Appl. 12 (2012), 121-133.

[33] F. Castella, *L^2 solutions to the Schrödinger-Poisson system : existence, uniqueness, time behavior and smoothing effects*, Math. Models Methods Appl. Sci. 8 (1997), 1051-1083.

[34] I. Catto, J. Dolbeault, O. Sánchez, J. Soler, *Existence of steady states for the Maxwell-Schrödinger-Poisson system : exploring the applicability of the concentration-compactness principle*, Math. Models Methods Appl. Sci. 23 (2013) 1915-1938.

[35] I. Catto, P. L. Lions, *Binding of atoms and stability of molecules in Hartree and Thomas-Fermi type theories. I. A necessary and sufficient condition for the stability of general molecular systems*, Comm. Partial Differential Equations 17 (1992), no. 7-8, 1051-1110.

[36] T. Cazenave, P. L. Lions, *Orbital Stability of Standing Waves for Some Non linear Schrödinger Equations*, Commun. Math. Phys. 85 (1982), no. 4, 549-561.

[37] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in mathematics 10, New York University, New York, (2003).

[38] J. Chen, Y. Li, Z.-Q. Wang, *Stability of standing waves for a class of quasilinear Schrödinger equations*, European Journal of Applied Math. 23 (2012), 611-633.

[39] Y. Cho, M. M. Fall, H. Hajaiej, P. A. Markowich, S. Trabelsi, *Orbital stability of standing waves of a class of fractional Schrödinger equations with a general Hartree-type integrand*, arXiv : 1307.5523v1 [math.AP] 21 Jul 2013.

[40] M. Colin, L. Jeanjean, *Solutions for quasilinear Schrödinger equation : a dual approach*, Nonlinear Analysis : T. M. A. 56 (2004), 213-226.

[41] M. Colin, L. Jeanjean, M. Squassina, *Stability and instability results for standing waves of quasilinear Schrödinger equations*, Nonlinearity 23 (2010), no. 6, 1353-1385.

[42] C. Chen, Y. Kuo, T. Wu, *Existence and multiplicity of positive solutions for the nonlinear Schrödinger-Poisson equations*, Proc. Roy. Soc. Edinburgh Sect. A 143 (2013), 745-764.

[43] T. D'Aprile, D. Mugnai, *Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations*, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), no. 5, 893-906.

[44] T. D'Aprile, D. Mugnai, *Non-existence results for the coupled Klein-Gordon-Maxwell equations*, Adv. Nonlinear Stud. 4 (2004), 307-322.

[45] P. D'Avenia, *Non-radially symmetric solutions of nonlinear Schrödinger equation coupled with Maxwell equations*, Adv. Nonlinear Stud. 2 (2002), 177-192.

[46] A. De Bouard, N. Hayashi, J. C. Saut, *Global existence of small solutions to a relativistic nonlinear Schrödinger equation*, Comm. Math. Phys. 189 (1997), 73-105.

[47] L. Dupaigne, Private communication.

[48] D. Fang, L. J. Xie, T. Cazenave, *Scattering for the focusing energy-subcritical NLS*, Sci. Chima Math. 54 (2011), 2037-2062.

[49] J. Fröhlich, E. H. Lieb, M. Loss, *Stability of Coulomb systems with magnetic fields. I. The one-electron atom*, Comm. Math. Phys. 104 (1986), no. 2, 251-270.

[50] X. D. Fang, A. Szulkin, *Multiple solutions for a quasilinear Schrödinger equation*, J. Diff. Equ. 254 (2012), 2015-2032.

[51] R. Fukuizumi, T. Ozawa, *Exponential decay of the solutions to nonlinear elliptic equations with potential*, Z. Angew. Math. Phys. 56 (2005), no. 6, 1000-1011.

[52] V. Georgiev, F. Prinari, N. Visciglia, *On the radiality of constrained minimizers to the Schrödinger-Poisson-Slater energy*, Ann. Inst. H. Poincaré Anal. Non. Linéaire 29, no. 3 (2012), 369-376.

[53] Nassif Ghoussoub, *Duality and Perturbation Methods in Critial Point Theory*, Cambridge Tracts in Mathematics, 107, Cambridge University Press, Cambridge, 1993.

[54] F. Gladiali, M. Squassina, *Uniqueness of ground states for a class of quasi-linear ellptic equations*, Adv. Nonlinear Anal. 1 (2012), 159-179.

[55] M. V. Goldman, M. Porkolab, *Upper hybrid solitons ans oscillating two-stream instabilities*, Physics of fuids 19 (1976), 872-881.

[56] H. Hajaiej and C. A. Stuart, *On the variational approach to the stability of standing waves for the nonlinear Schrödinger equation*, Adv. Nonlinear Stud. 4 (2004), no. 4, 469-501.

[57] H. Hartmann and W. J. Zakrzewski, *Electrons on hexagonal lattices and applications to nanotubes*, Phys. Rev. B 68 (2003), 184-302.

[58] R. W. Hasse, *A general method for the solution of nonlinear soliton and kink Schrödinger equations*, Z. Physik B 37 (1980), 83-87.

[59] J. Hirata, N. Ikoma, K. Tanaka, *Nonlinear scalar field equations in $\mathbb{R}^N$: mountain pass ans symmetric moutain pass approaches*, Top. Meth. Nonlin. Anal. 35 (2010), 253-276.

[60] J. Holmer, S. Roudenko, *On blow-up solutions to the 3D cubic Nonlinear Schrödinger Equation*, App. Math. Res. Express. AMRX, (2007).

[61] Y. Huang, Z. Liu, Y. Wu, *Existence of Prescribed-Norm Solutions for a Class of Schrödinger-Poisson Equation*, Abstract and Applied Analysis, to appear.

[62] I. Ianni, *Sign-changing radial solutions for the Schrödinger-Poisson-Slater problem*, Topol. Methods Nonlinear Anal. 41 (2013), no. 2, 365-385.

[63] I. Ianni, D. Ruiz, *Ground and bound states for a static Schrödinger-Poisson-Slater problem*, Comm. Comp. Math. 14 (2012), no. 1, DOI : 10.1142/S0219199712500034.

[64] L. Jeanjean, *Existence of solutions with prescribed norm for semilinear elliptic equations*, Nonlinear Analysis : T. M. A. 28 (1997), no. 10, 1633-1659.

[65] L. Jeanjean, T-J. Luo, *Sharp non-existence results of prescribed L^2-norm solutions for some class of Schrödinger-Poisson and quasi-linear equations*, Z. Angew. Math. Phys. 64 (2013), 937-954.

[66] L. Jeanjean, T-J. Luo, Z.-Q. Wang, *Multiple normalized solutions for quasi-linear Schrödinger equations*, preprint.

[67] L. Jeanjean, M. Squassina, *An approach to minimization under a constraint : the added mass technique*, Calc. Var. Partial Differential Equations 41 (2011), no. 3-4, 511-534.

[68] M. Kaminaga, M. Ohta, *Stability of standing waves for nonlinear Schrödinger equation with attractive delta potential and repulsive nonlinearity*, Saitama Math. J. 26 (2009), 39-48.

[69] T. Kato, *Growth properties of the solutions of the reduced wave equation with a variable coefficient*, Comm. Pure Appl. Math. 12 (1959), 403-425.

[70] H. Kikuchi, *On the existence of solutions for elliptic system related to the Maxwell-Schrödinger equations*, Nonlinear. Anal. 67 (2007), 1445-1456.

[71] H. Kikuchi, *Existence and stability of standing waves for Schrödinger-Poisson-Slater equation*, Adv. Nonlinear Stud. 7 (2007), no. 3, 403-437.

[72] H. Kikuchi, *Existence and orbital stability of the standing waves for nonlinear Schrödinger equations via the variational method*, Doctoral Thesis (2008).

[73] W. Kohn and J. L. Sham, *Self-consistent equations including exchange and correction effects*, Phys. Rev. 140 (1965), 1133-1138.

[74] S. Kurihara, *Large-Amplitude Quasi-Solitons in Superfluid Films*, J. Phys. Soc. Jpn. 50 (1981), 3262-3267.

[75] M. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^N$*, Arch. Rational Mech. Anal. 105 (1989), 243-266.

[76] E. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Appl. Math. 57 (1977), no. 2, 93-105.

[77] E. H. Lieb, *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. math. 74, 441-448 (1983).

[78] E. H. Lieb, *Thomas-Fermi and related theories of atoms and molecules*, Rev. Modern Phys. 53 (1981), 603-641.

[79] E. H. Lieb, *Thomas-Fermi Theory*, Kluwer Encyclopedia of Mathematics, Supplement, Vol. II (2000), 311-313.

[80] E. H. Lieb and M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.

[81] E. H. Lieb, B. Simon, *The Thomas-Fermi theory of atoms, molecules, and solids*, Advances in Math. 23 (1977), no. 1, 22-116.

[82] E. H. Lieb, H. T. Yau, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys. 112 (1987), 147-174.

[83] P. L. Lions, *The concentration-compactness principle in the Calculus of Variation. The locally compact case, part I and II*, Ann. Inst. H. Poincaré Anal. Non. Linéaire 1 (1984), no. 2, 109-145 and no. 4, 223-283.

[84] P. L. Lions, *Solutions of Hartree-Fock Equations for Coulomb Systems*, Comm. Math. Phys. 109 (1987), no. 1, 33-97.

[85] A. G. Litvak, A. M. Sergeev, *One dimensional collapse of plasma waves*, JETP, Letters 27 (1978), 517-520.

[86] X.-Q. Liu, J.-Q. Liu, Z.-Q. Wang, *Quasilinear elliptic equations via perturbation method*, Proc. Amer. Math. Soc. 141 (2013), no. 1, 253-263.

[87] J. Liu, Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations I*, Proc. Amer. Math. Soc. 131 (2003), 441-448.

[88] J. Liu, Y. Wang, Z.-Q. Wang, *Soliton solutions for quasilinear Schrödinger equations II*, J. Differential Equations 187 (2003), 473-493.

[89] J. Liu, Y. Wang, Z.-Q. Wang, *Solutions for quasilinear Schrödinger equations via the Nehari method*, Comm. P.D.E. 29 (2004), 879-901.

[90] T-J. Luo, *Multiplicity of normalized solutions for a class of nonlinear Schrödinger-Poisson-Slater equations*, preprint.

[91] V. G. Makhankov, V. K. Fedynanin, *Non-linear effects in quasi-one-dimensional models of condensed matter theory*, Physics Reports 104 (1984), 1-86.

[92] N. J. Mauser, *The Schrödinger-Poisson-Xα equation*, Appl. Math. Lett. 14 (2001), no. 6, 759-763.

[93] F. Merle, P. Raphael, *On universality of blow up profile for L^2-critical nonlinear Schrödinger equation*, Invent. Math. 156 (2004), 565-672.

[94] F. Merle, P. Raphael, J. Szeftel, *Stable self-similar blow-up dynamics for slightly L^2 super-critical NLS equations*, Geom. Funct. Anal. 20 (2010), 1028-1071.

[95] V. Moroz, J. Van Schaftingen, *Existence of groundstates for a class of nonlinear Choquard equations*, Trans. Amer. Math. Soc. to appear.

[96] M. Poppenberg, *On the local well posedness of quasi-linear Schröodinger equations in arbitrary space dimension*, J. Differential Equations 172 (2001), 83-115.

[97] M. Poppenberg, K. Schmitt, Z.-Q. Wang, *On the existence of soliton solutions to quasilinear Schrödinger equations*, Calc. Var. and P.D.E. 14 (2002), 329-344.

[98] B. Ritchie, *Relativistic self-focusing and channel formation in laser-plasma interactions*, Phys. Rev. E 50 (1994), 687-689.

[99] D. Ruiz, *The Schrödinger-Poisson equation under the effect of a nonlinear local term*, J. Funct. Anal. 237 (2006), no. 2, 655-674.

[100] D. Ruiz, *On the Schrödinger-Poisson-Slater System : Behavior of Minimizers, Radial and Nonradial Cases*, Arch. Rational Mech. Anal. 198 (2010), no. 1, 349-368.

[101] D. Ruiz, G. Siciliano, *Existence of ground states for a modified nonlinear Schrödinger equation*, Nonlinearity 23 (2010), 1221-1233.

[102] O. Sánchez, J. Soler, *Long-time dynamics of the Schrödinger-Poisson-Slater system*, J. Statist. Phys. 114 (2004), no. 1-2, 179-204.

[103] A. Selvitella, *Uniqueness and nondegeneracy of the ground state for a quasilinear Schrödinger equation with a small parameter*, Nonlinear Analysis : T. M. A. 74 (2011), 1731-1737.

[104] G. Siciliano, *Multiple positive solutions for a Schrödinger-Poisson-Slater system*, J. Math. Anal. Appl. 365 (2010), no. 1, 288-299.

[105] G. Siciliano, *A minimization problem for the nonlinear Schrödinger-Poisson type equation*, São Paulo J. Math. Sci. 5 (2011), no. 2, 149-173.

[106] J. C. Slater, *A Simplification of the Hartree-Fock Method*, Phys. Rev. 81 (1951), 385-390.

[107] M. Struwe, *Variational Methods.* Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Third Edition, 34. Springer-Verlag, Berlin, 1996.

[108] Z. Wang, H-S. Zhou, *Positive solution for a nonlinear stationary Schrödinger-Poisson system in* $\mathbb{R}^3$, Discrete Contin. Dyn. Syst. 18 (2007), no. 4, 809-816.

[109] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Comm. Math. Phys. 87 (1983), 567-576.

[110] M. Willem, *Minimax Theorems.* Progress in Nonlinear Differential Equations and their Applications, 24. Birkhäuser Boston, Inc., Boston, MA, 1996.

[111] D. Wu, *Existence and stability of standing waves for nonlinear fractional Schrödinger equations with Hartree type nonlinearity*, J. Math. Anal. Appl. to appear.

[112] L. Zhao, F. Zhao, *On the existence of solutions for the Schrödinger-Poisson equations*, J. Math. Anal. Appl. 346 (2008), no. 1, 155-169.